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FORMULAS AND TABLES FOR THE CALCULATION OF MUTUAL AND SELF-INDUCTANCE [Revised]

BY

E. B. ROSA, Chief Physicist
and
F. W. GROVER, Associate Physicist
Bureau of Standards

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FORMULAS AND TABLES FOR THE CALCULATION OF MUTUAL AND SELF-INDUCTANCE

By Edward B. Rosa and Frederick W. Grover

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INTRODUCTION

A great many formulas have been given for calculating the mutual and self-inductance of the various cases of electrical circuits occuring in practice. Some of these formulas have subsequently been shown to be wrong, and of those which are correct and applicable to any given case there is usually a choice, because of the greater accuracy or greater convenience of one as compared with the others. For the convenience of those having such calculations to make we have brought together in this paper all the formulas with which we are acquainted which are of value in the calculation of mutual and self-inductance, particularly in nonmagnetic circuits where the frequency of the current is low enough to assure sensibly uniform distribution of current. In the last section some formulas are given for the variation of the self-inductance and resistance with the frequency. A considerable number of formulas which have been shown to be unreliable or which have been replaced by others that are less complicated or more accurate have been omitted, although in most cases we have given references to such omitted formulas. Where several formulas are applicable to the same case we have pointed out the especial advantage of each and indicated which one is best adapted to precision work.

In the second part of each section of the paper we give a number of examples to illustrate and test the formulas. We have given the work in many cases in full to serve as a guide in such calculations in order to make the formulas as useful as possible to students and others not familiar with such calculations, and also to facilitate the work of checking up the results by anyone going over the subject. We have been impressed with the importance of this in reading the work of others.

In the appendix to the paper are a number of tables that will be found useful in numerical calculations of inductance.

In most cases we have given the name of the author of a formula in connection with the formula. This is not merely for the sake of historical interest, or to give proper credit to the authors, but also because we have found it helpful to distinguish in this way the various formulas instead of denoting each merely by a number. The formulas of sections 8 and 9, which are taken largely from a paper by one of the present authors,[1] are, however, not so designated, although the authorship of those that are not new is indicated where known.

This paper includes practically all the matter contained in the 1907 paper under the same title by Rosa and Cohen, but in addition to a thorough revision in which some errors are corrected and some formulas extended, a large amount of new matter has been added both in the body of the paper and in the tables. We shall be grateful to anyone detecting any errors either in formulas or tables if he communicates the same to us.

1. MUTUAL INDUCTANCE OF TWO COAXIAL CIRCLES

MAXWELL'S FORMULAS IN ELLIPTIC INTEGRALS

The first and most important of the formulas for the mutual inductance of coaxial circles is the formula in elliptic integrals given by Maxwell: [2]

$$M=4\pi\sqrt{Aa}\left\{\left(\frac{2}{k}-k\right)F-\frac{2}{k}E\right\} \qquad [1]$$

in which A and a are the radii of the two circles, d is the distance between their centers, and

$$k=\frac{2\sqrt{Aa}}{\sqrt{(A+a)^2+d^2}}=\sin\gamma=\frac{\sqrt{r_1^2-r_2^2}}{r_1}$$

F and E are the complete elliptic integrals of the first and second kind, respectively, to modulus k. Their values may be obtained from the tables of Legendre (see Tables XII and XIII in the Appendix), or the values of $M\div 4\pi\sqrt{Aa}$ may be obtained from Table I in the appendix of this paper, the values of γ being the argument.

---

[1] This Bulletin, 4, p 301; 1907.

[2] Electricity and Magnetism, Vol. II, § 701.

The notation of Maxwell is slightly altered in the above expressions in order to bring it into conformity with the formulas to follow.

Formula (1) is an absolute one, giving the mutual inductance of two coaxial circles of any size at any distance apart. If the two circles have equal or nearly equal radii, and are very near each other, the quantity k will be very nearly equal to unity and γ will be near to 90°. Under these circumstances it may be difficult to obtain a sufficiently exact value of F and E from the tables, as the quantities are varying rapidly and the tabular differences are very large. Under such circumstances the following formula, also given by Maxwell[2] (derived by means of Landen's transformation), is more suitable:

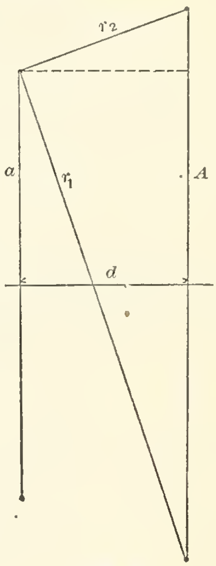

Fig. 1

$$M = 8\pi\frac{\sqrt{Aa}}{\sqrt{k_1}}\left\{F_1 - E_1\right\} \qquad [2]$$

in which F_1 and E_1 are complete elliptic integrals to modulus k_1, and

$$k_1 = \frac{r_1 - r_2}{r_1 + r_2} = \sin\gamma_1 = \frac{4Aa}{(r_1 + r_2)^2}$$

r_1 and r_2 are the greatest and least distances of one circle from the other (Fig. 1); that is,

$$r_1 = \sqrt{(A+a)^2 + d^2}$$
$$r_2 = \sqrt{(A-a)^2 + d^2}$$

The new modulas k_1 differs from unity more than k, hence γ_1 is not so near to 90° as γ and the values of the elliptic integrals can be taken more easily from the tables than when using formula (1) and the modulus k.

Another way of avoiding the difficulty when k is nearly unity is to calculate the integrals F and E directly, and thus not use the

tables of elliptic integrals, expanding F and E in terms of the complementary modulus k', where $k'=\sqrt{1-k^2}$. k' may usually be more accurately calculated by the formula $k'=\frac{r_2}{r_1}$. The expressions for F and E are very convergent when k' is small.

$$
\begin{aligned}
F=\log\frac{4}{k'}&+\frac{1^2}{2^2}k'^2\left(\log\frac{4}{k'}-\frac{2}{1.2}\right)\\
&+\frac{1^2}{2^2}\frac{3^2}{4^2}k'^4\left(\log\frac{4}{k'}-\frac{2}{1.2}-\frac{2}{3.4}\right)\\
&+\frac{1^2}{2^2}\frac{3^2}{4^2}\frac{5^2}{6^2}k'^6\left(\log\frac{4}{k'}-\frac{2}{1.2}-\frac{2}{3.4}-\frac{2}{5.6}\right)\\
&+\frac{1^2}{2^2}\frac{3^2}{4^2}\frac{5^2}{6^2}\frac{7^2}{8^2}k'^8\left(\log\frac{4}{k'}-\frac{2}{1.2}-\frac{2}{3.4}-\frac{2}{5.6}-\frac{2}{7.8}\right)\\
&+\ldots\ldots\ldots\ldots
\end{aligned}
\qquad [3]
$$

$$
\begin{aligned}
E=1&+\frac{1}{2}k'^2\left(\log\frac{4}{k'}-\frac{1}{1.2}\right)\\
&+\frac{1^2}{2^2}\frac{3}{4}k'^4\left(\log\frac{4}{k'}-\frac{2}{1.2}-\frac{1}{3.4}\right)\\
&+\frac{1^2}{2^2}\frac{3^2}{4^2}\frac{5}{6}k'^6\left(\log\frac{4}{k'}-\frac{2}{1.2}-\frac{2}{3.4}-\frac{1}{5.6}\right)\\
&+\frac{1^2}{2^2}\frac{3^2}{4^2}\frac{5^2}{6^2}\frac{7}{8}k'^8\left(\log\frac{4}{k'}-\frac{2}{1.2}-\frac{2}{3.4}-\frac{2}{5.6}-\frac{1}{7.8}\right)\\
&+\ldots\ldots\ldots\ldots
\end{aligned}
$$

The equations (3) are very convergent for $k'<0.1$, ($k\geqq 0.995$), and satisfactory accuracy will be attained down to $k=0.985$, thus covering the range of values for which interpolation in Legendre's tables becomes difficult.

For values of k near 0.985 it is perhaps more accurate to calculate M from elliptic integrals F_0 and E_0 with a modulus k_0 greater than k. The modulus k_0' which is complementary to k_0 is smaller than k', and the values of F_0 and E_0 calculated from the series formulas (3) putting k_0' in place of k' converge more rapidly than the values of F and E when calculated by the same series formulas. The formula for making the transformation is not quite so simple as (2). It is most conveniently written

$$\left.\begin{aligned}
M &= 4\pi\sqrt{Aa}\left[\frac{F_0}{k(1+k)} - \frac{(1+k)}{k}E_0\right] \\
k_0{}' &= \frac{1-k}{1+k} = \frac{k'^2}{(1+k)^2} \\
k &= \frac{2\sqrt{Aa}}{\sqrt{(A+a)^2+d^2}} \qquad k' = \frac{\sqrt{(A-a)^2+d^2}}{\sqrt{(A+a)^2+d^2}}
\end{aligned}\right\} \qquad [4]$$

When the distance between the circles is large, formula (1) becomes unsuitable for calculation for two reasons, (*a*) because γ falls outside the range of Table XIII and (*b*) because the quantity $\left(\frac{2}{k} - k\right)F - \frac{2}{k}E$ comes out as the small difference of two large quantities. The use of formula (4) overcomes the first objection, but makes the matter still worse as far as the second is concerned. We may, however, express (1) in terms of a series by means of the well known expressions of Wallis[3]

$$F = \frac{\pi}{2}\left[1 + \sum_{1}^{\infty}\left\{\frac{1\cdot3\cdot5\cdot\cdots\cdot(2n-1)}{2\cdot4\cdot6\cdot\cdots\cdot 2n}\right\}^2 k^{2n}\right]$$

$$E = \frac{\pi}{2}\left[1 - \sum_{1}^{\infty}\left\{\frac{1\cdot3\cdot5\cdot\cdots\cdot(2n-1)}{2\cdot4\cdot6\cdot\cdots\cdot 2n}\right\}^2 \frac{k^{2n}}{(2n-1)}\right]$$

Substituting these values in (1) we find

$$M = \frac{\pi^2 k^3}{4}\sqrt{Aa}\left[1 + \frac{3}{4}k^2 + \frac{75}{128}k^4 + \frac{245}{512}k^6 + \cdots\right] \qquad [5]$$

the general term in the brackets being

$$\left(\frac{3\cdot5\cdot7\cdots(2n+1)}{4\cdot6\cdot8\cdots(2n+2)}\right)^2\frac{(2n+2)}{(n+2)}k^{2n} = \left[\frac{3\cdot5\cdot7\cdots(2n+1)}{4\cdot6\cdot8\cdots 2n}\right]^2\frac{k^{2n}}{(n+2)(2n+2)}$$

For values of k up to 0.1 ($\gamma = 5°.7$) the series (5) is very convergent, and may be used for values of k up to 0.2 ($\gamma = 11°.5$) without serious labor. In the latter case and for still larger values of k, we may calculate M in terms of the smaller modulus k_1 of formula (2). This last expression becomes on expansion

[3] Greenhill's "Elliptic Functions," pp. 9, 176.

$$M = 2\pi^2 k_1^{\frac{3}{2}} \sqrt{Aa} \left[1 + \frac{3}{8} k_1^2 + \frac{15}{64} k_1^4 + \frac{175}{1024} k_1^6 + \cdots \right] \quad [6]$$

the general term in the brackets being

$$\left(\frac{n+1}{2n+1} \right) \left[\frac{3 \cdot 5 \cdot 7 \cdots (2n+1)}{4 \cdot 6 \cdot 8 \cdots (2n+2)} \right]^2 k^{2n}$$

The series (6) converges more rapidly than (5), and may be used with ease for values of k_1 as great as $\frac{1}{4}$, ($\gamma_1 = 14°.5$), which corresponds to $k = 0.8$, ($\gamma = 53°.2$).

To recapitulate—

(1) For values of k between zero and 0.2 use (5).

(2) For values of k a little larger and up to 0.8 use (6).

(3) For values of k between about 0.7 and 0.985 the elliptic integrals in (1) may be conveniently taken by interpolation from Legendre's tables or from Table XIII.

(4) For values of k greater than about 0.7 we may use (4).

(5) For values of k greater than about 0.985 we may use (3).

It will be thus seen that the formulas overlap, so that it will be possible in every case to calculate the mutual inductance by at least two different formulas, the less accurate serving as a check on the more accurate.

The choice of formulas is considered more in detail on page 19.

WEINSTEIN'S FORMULA

Weinstein[4] gives an expression for the mutual inductance of two coaxial circles, in terms of the complementary modulus k' used in the preceding series (3). Substituting in equation (1) the values of F and E given above we have Weinstein's equation, which is as follows:

$$M = 4\pi\sqrt{Aa} \left\{ \left(1 + \frac{3}{4} k'^2 + \frac{33}{64} k'^4 + \frac{107}{256} k'^6 + \frac{5913}{16384} k'^8 + \cdots \right) \left(\log \frac{4}{k'} - 1 \right) - \left(1 + \frac{15}{128} k'^4 + \frac{185}{1536} k'^6 + \frac{7465}{65536} k'^8 + \cdots \right) \right\} \quad [7]$$

[4] Wied. Ann., **21**, p. 344; 1884.

This expression is rapidly convergent when k' is small, and hence will give an accurate value of M when the circles are near each other. Otherwise formula (1) may be more suitable.

NAGAOKA'S FORMULAS

Nagaoka[5] has given formulas for the calculation of the mutual inductance of coaxial circles, without the use of tables of elliptic integrals. These formulas make use of Jacobi's q-series, which is very rapidly convergent. The first is to be used when the circles are not near each other, the second when they are near each other. Either may be employed for a considerable range of distances between the extremes, although the first is more convenient. The *first* formula is as follows:

$$M = 16\pi^2\sqrt{Aa}\, q^{\frac{3}{2}}(1+\epsilon)$$
$$= 4\pi\sqrt{Aa}\{4\pi q^{\frac{3}{2}}(1+\epsilon)\} \qquad [8]$$

where A and a are the radii of the two circles. The correction term ϵ can be neglected when the circles are quite far apart.

$$\epsilon = 3q^4 - 4q^6 + 9q^8 - 12q^{10} + \cdots\cdots$$
$$q = \frac{l}{2} + 2\left(\frac{l}{2}\right)^5 + 15\left(\frac{l}{2}\right)^9 + \cdots\cdots$$
$$l = \frac{1-\sqrt{k'}}{1+\sqrt{k'}} \qquad k' = \frac{r_2}{r_1} = \frac{\sqrt{(A-a)^2+d^2}}{\sqrt{(A+a)^2+d^2}}$$

d being the distance between the centers of the circles, and k' the complementary modulus occurring in equations (3) and (7).

Nagaoka's *second* formula is as follows:

$$M = 4\pi\sqrt{Aa}.\frac{1}{2(1-2q_1)^2}\left\{[1+8q_1(1-q_1+4q_1^2-\cdots\cdots)]\log\frac{1}{q_1} - 4\right\} \quad [9]$$

$$= 4\pi\sqrt{Aa}.\frac{1}{2(1-2q_1)^2}\left\{[1+8q_1-8q_1^2+\epsilon_1]\log\frac{1}{q_1} - 4\right\}$$

$$q_1 = \frac{l_1}{2} + 2\left(\frac{l_1}{2}\right)^5 + 15\left(\frac{l_1}{2}\right)^9 + \cdots$$
$$l_1 = \frac{1-\sqrt{k}}{1+\sqrt{k}} \qquad k = \frac{2\sqrt{Aa}}{\sqrt{(A+a)^2+d^2}} \qquad [9a]$$
$$\epsilon_1 = 32q_1^3 - 40q_1^4 + 48q_1^5 - \cdots\cdots$$
$$-\epsilon_1' = 8q_1^2 - \epsilon_1$$

[5] Phil. Mag., 6, p. 19; 1903. Recently a third expression has been found by Nagaoka (Tokyo Math. Phys. Soc., 6, p. 10; 1911). (See p. 187 below.)

k is the modulus of equation (1), but is employed here to obtain the value of the q-series instead of the values of the elliptic integrals employed in (1). This formula is ordinarily simpler in use than it appears, because some of the terms in the expressions above are usually negligible. For a third formula see page 187.

Nagaoka has recently published[6] tables which materially reduce the labor of calculation with these formulas. These are reproduced as Tables XV and XVI of the appendix. From Table XV we obtain directly the small difference $q-\frac{l}{2}$ or $q_1-\frac{l_1}{2}$ with q or q_1 as argument. The same table gives also the corresponding values of ϵ and $\log_{10}(1+\epsilon)$ for use in the formula (8).

To calculate q or q_1 we enter the table with $\frac{l}{2}$ or $\frac{l_1}{2}$ as argument. The difference corresponding in the table when added to $\frac{l}{2}$ or $\frac{l_1}{2}$ gives the value of q or q_1 to a first approximation. This will be sufficient except for the larger values of q or q_1 which are tabulated here. For these it is sometimes necessary to use this first approximation as argument to obtain a more accurate value of q or q_1.

Table XVI gives the values of ϵ_1 and $-\epsilon_1'$ for different values of q_1 and is useful in calculations with formula (9).

For circles at some distance from one another q becomes small, and the expression for l given above becomes inconvenient, because k' is so nearly equal to unity. In this case we may calculate l from the somewhat more complicated expression

$$l=\frac{k^2}{(1+k')(1+\sqrt{k'})^2}$$

the values of k and k' being calculated from the formulas already given. The same applies to the calculation of l_1 in formula (9*a*), when the circles are very near together, and consequently q_1 is very small. For this case we use the expression

$$l_1=\frac{k'^2}{(1+k)(1+\sqrt{k})^2}$$

[6] Jour. of Coll. of Sci. Tokyo, vol. 27, art. 6; 1909.

MAXWELL'S SERIES FORMULA

Maxwell[7] obtained an expression for the mutual inductance between two coaxial circles in the form of a converging series which is often more convenient to use than the elliptical integral formula, and when the circles are nearly of the same radii and relatively near each other the value given is generally sufficiently exact. In the following formula a is the smaller of the two radii, c is their difference, $A-a$, d is the distance apart of the circles as before, and $r=\sqrt{c^2+d^2}$. The mutual inductance is then

$$M=4\pi a\left\{\log\frac{8a}{r}\left(1+\frac{c}{2a}+\frac{c^2+3d^2}{16a^2}-\frac{c^3+3cd^2}{32a^3}+\cdots\right)\right.$$
$$\left.-\left(2+\frac{c}{2a}-\frac{3c^2-d^2}{16a^2}+\frac{c^3-6cd^2}{48a^3}-\cdots\right)\right\} \qquad [10]$$

When c and d are small compared with a, we have for an approximate value of the mutual inductance the following simple expression:[8]

$$M=4\pi a\left\{\log\frac{8a}{r}-2\right\} \qquad [11]$$

When the two radii are equal, as is often the case in practice, the equation (10) is somewhat simplified, as follows:

$$M=4\pi a\left\{\log\frac{8a}{d}\left(1+\frac{3d^2}{16a^2}\right)-\left(2+\frac{d^2}{16a^2}\right)\right\} \qquad [12]$$

The above formulas (10) and (12) are sufficiently exact for very many cases, the terms omitted in the series being unimportant when $\frac{c}{a}$ and $\frac{d}{a}$ are small. For example, if $\frac{d}{a}$ is 0.1, the largest term neglected in (12) is less than two parts in a million. If, however, $d=a$, this term will be more than one per cent, and the formula will be quite inexact.

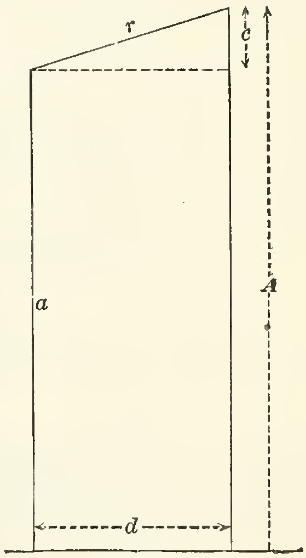

Fig. 2

[7] Electricity and Magnetism, Vol. II, § 705.

[8] This is equivalent to the approximate formula given by Wiedemann, $M=4\pi a\left\{\log\frac{2l}{c}-2.45\right\}$, where l is the circumference of the smaller circle and c is the same as r above.

Coffin[9] has extended Maxwell's formula (12) for two equal circles by computing three additional terms for each part of the expression. This enables the mutual inductance to be computed with considerable exactness up to $d=a$. Formula (1) is exact, as stated above, for all distances, and either it or (8) should be used in preference to (13) when d is large. Coffin's formula is as follows:

$$M=4\pi a\left\{\log\frac{8a}{d}\left(1+\frac{3d^2}{16a^2}-\frac{15d^4}{8\times 128a^4}+\frac{35d^6}{128^2a^6}-\frac{1575d^8}{2\times 128^3a^8}+\cdots\right)\right.$$
$$\left.-\left(2+\frac{d^2}{16a^2}-\frac{31d^4}{16\times 128a^4}+\frac{247d^6}{6\times 128^2a^6}-\frac{7795d^8}{8\times 128^3a^8}+\cdots\right)\right\} \quad [13]$$

We have extended Maxwell's formula (10) for unequal circles as follows:[10]

$$M=4\pi a\left\{\log\frac{8a}{r}\left(1+\frac{c}{2a}+\frac{c^2+3d^2}{16a^2}-\frac{c^3+3cd^2}{32a^3}+\frac{17c^4+42c^2d^2-15d^4}{1024a^4}\right.\right.$$
$$\left.-\frac{19c^5+30c^3d^2-45cd^4}{2048a^5}+\cdots\right)-\left(2+\frac{c}{2a}-\frac{3c^2-d^2}{16a^2}+\frac{c^3-6cd^2}{48a^3}\right.$$
$$\left.\left.+\frac{19c^4+534c^2d^2-93d^4}{6144a^4}-\frac{379c^5+3030c^3d^2-1845cd^4}{61440a^5}\right)\right\} \quad [14]$$

Nagaoka[11] has confirmed this extension by expanding formula (9). He carried out the expansion, however, no further than terms in $\frac{c^4}{a^4}$ and $\frac{d^4}{a^4}$.

When $c=0$, this gives the first part of series (13). When $d=0$, the case of two circles in the same plane, with radii a and $a+c$, we have

$$M=4\pi a\left\{\log\frac{8a}{c}\left(1+\frac{c}{2a}+\frac{c^2}{16a^2}-\frac{c^3}{32a^3}+\frac{17c^4}{1024a^4}-\frac{19c^5}{2048a^5}+\cdots\right)\right.$$
$$\left.-\left(2+\frac{c}{2a}-\frac{3c^2}{16a^2}+\frac{c^3}{48a^3}+\frac{19c^4}{6144a^4}-\frac{379c^5}{61440a^5}+\cdots\right)\right\} \quad [15]$$

---

[9] J. G. Coffin, this Bulletin, **2**, p. 113; 1906.

[10] This Bulletin, **2**, p. 364; 1906.

[11] Loc. cit., p. 11.

These formulas (14) and (15) give the mutual inductance with great precision when the circles are not too far apart. The degree of convergence, of course, indicates approximately in any case the accuracy of the result.

HAVELOCK'S FORMULA

In 1908 Havelock[12] published a paper in which the calculation of mutual and self-inductance is made to depend on the evaluation of certain definite integrals of Bessel functions of the form $\int_0^\infty e^{-p\mu} J_1(\mu) J_1(\lambda\mu)\mu^{-n} d\mu$. These he expands in the form of series, which fall into two classes, those suitable for small values of p, and those suitable for large values of p. In the case of the latter, he gives the expressions for the general terms of the series, so that these may be extended as far as desired. In the case of the former only a few terms are given, and the derivation of further terms is very tedious.

He considers first the mutual inductance of two coaxial circles, and points out that the solution may be made to depend on either of two of his integrals. He does not, however, write out the formulas. It is a simple matter to carry out the necessary substitutions, and we find for circles near one another

$$M = 4\pi\sqrt{Aa}\left\{\left[1 + \frac{3}{16}\left(\frac{r^2}{Aa}\right) - \frac{15}{1024}\left(\frac{r^2}{Aa}\right)^2 + \cdots\right]\log\frac{8\sqrt{Aa}}{r} - \left(2 + \frac{1}{16}\left(\frac{r^2}{Aa}\right) - \frac{31}{2048}\left(\frac{r^2}{Aa}\right)^2 + \cdots\right)\right\}$$

This expression bears some resemblance to Maxwell's series formula (10); it is, however, simpler for use in calculation. To obtain the coefficients of further terms by Havelock's process would require a good deal of labor. We notice, however, that, putting $A = a$, the formula becomes the same as Coffin's formula (13) for equal circles. We may evidently, therefore, use the coefficients of the higher order terms in Coffin's formula to obtain an extension of the above, and find the expression

[12] Phil. Mag., **15**, p. 332; 1908.

$$M=4\pi\sqrt{Aa}\left\{\left[1+\frac{3}{16}\alpha-\frac{15}{1024}\alpha^2+\frac{35}{128^2}\alpha^3-\frac{1575}{2\times128^3}\alpha^4+\cdots\right]\log\frac{8\sqrt{Aa}}{r}\right.$$
$$\left.-\left(2+\frac{1}{16}\alpha-\frac{31}{2048}\alpha^2+\frac{247}{6\times128^2}\alpha^3-\frac{7795}{8\times128^3}\alpha^4+\cdots\right)\right\} \qquad [16]$$

where

$$r^2=c^2+d^2 \qquad \alpha=\frac{a}{A}\frac{r^2}{a^2}$$

The expression thus extended[13] gives very accurate results for values of d almost as great as the radius a. For a given degree of convergence it requires only half as many terms to be calculated as does formula (14), and is much easier to calculate.

The second formula derived from Havelock's paper is not so generally useful, being rapidly convergent only for values of d greater than about $5A$. It is

$$\begin{aligned}M=\frac{2\pi^2a^2A^2}{d^3}\Bigg[1&-\frac{3}{2}\left(1+\frac{a^2}{A^2}\right)\frac{A^2}{d^2}+\frac{15}{8}\left(1+3\frac{a^2}{A^2}+\frac{a^4}{A^4}\right)\frac{A^4}{d^4}\\&-\frac{35}{16}\left(1+6\frac{a^2}{A^2}+6\frac{a^4}{A^4}+\frac{a^6}{A^6}\right)\frac{A^6}{d^6}\\&+\frac{315}{128}\left(1+10\frac{a^2}{A^2}+20\frac{a^4}{A^4}+10\frac{a^6}{A^6}+\frac{a^8}{A^8}\right)\frac{A^8}{d^8}\\&-\frac{693}{256}\left(1+15\frac{a^2}{A^2}+50\frac{a^4}{A^4}+50\frac{a^6}{A^6}+15\frac{a^8}{A^8}+\frac{a^{10}}{A^{10}}\right)\frac{A^{10}}{d^{10}}+\cdots\Bigg]\end{aligned} \qquad [17]$$

For the case of $d=10A$ and a as great as A, only three terms have to be calculated to obtain M to about one part in a million, and for a smaller value of $\frac{a}{A}$ the convergence would be more rapid still.

MATHY'S FORMULA

In an interesting paper in the Journal de Physique for 1901,[14] E. Mathy obtained a formula for the mutual inductance of two circles,

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[13] Mr. T. J. Bromwich, of Cambridge, England, has recently communicated to us the same formula, without giving the proof, including however terms no higher than those in α^3.

[14] Jour. de Phys., **10**, p. 33; 1901.

in which the elliptic integral of the third kind, on which the mutual inductance depends, is expanded in a manner still different from that adopted in any of the preceding cases. It is expressed in terms of hypergeometric series involving the absolute invariant J of the Weierstrassian $\wp$ function. The final expression as found by Mathy is incorrect as regards the coefficients of the hypergeometric series. The corrected expression,[15] using the notation of this paper, is as follows:

$$\frac{M}{4\pi}=\Bigg[\frac{x^2}{(x^4+12A^2a^2)^{\frac{1}{4}}}\left\{\frac{P}{\sqrt[4]{27}}F\left(\frac{1}{12},\frac{5}{12},\frac{1}{2},\frac{J-1}{J}\right)-\frac{Q}{6\sqrt[4]{3}}\sqrt{\frac{J-1}{J}}\right.$$
$$\left.F\left(\frac{7}{12},\frac{11}{12},\frac{3}{2},\frac{J-1}{J}\right)\right\}-(x^4+12A^2a^2)^{\frac{1}{4}}\left\{\frac{Q}{\sqrt[4]{3}}F\left(-\frac{1}{12},\frac{7}{12},\frac{1}{2},\frac{J-1}{J}\right)\right.$$
$$\left.+\frac{P}{6\sqrt[4]{27}}\sqrt{\frac{J-1}{J}}F\left(\frac{5}{12},\frac{13}{12},\frac{3}{2},\frac{J-1}{J}\right)\right\}\Bigg] \qquad [18]$$

where

$$x^2=a^2+A^2+d^2$$

$$P=1.311028777 \cdots$$

$$Q=0.599070117 \cdots$$

$$\sqrt{\frac{J-1}{J}}=\frac{1-36\left(\frac{aA}{x^2}\right)^2}{\left[1+12\left(\frac{aA}{x^2}\right)^2\right]^{\frac{3}{2}}}$$

and

$$F(\alpha,\beta,\gamma,z)=1+\frac{\alpha\beta}{1\cdot\gamma}z+\frac{\alpha(\alpha+1)\cdot\beta(\beta+1)}{1\cdot2\cdot\gamma(\gamma+1)}z^2$$
$$+\frac{\alpha(\alpha+1)(\alpha+2)\cdot\beta(\beta+1)(\beta+2)}{1\cdot2\cdot3\cdot\gamma(\gamma+1)(\gamma+2)}z^3+\cdots$$

This formula is by no means so formidable to use as might be expected, since the constants which enter and the coefficients in the hypergeometric series may be calculated once for all. Using seven place logarithms we find

[15] Grover, this Bulletin, 6, p. 489; 1910.

$$\log_{10}\frac{P}{\sqrt[4]{27}}=9.7597712 \qquad \log_{10}\frac{P}{6\sqrt[4]{27}}=8.9816199$$

$$\log_{10}\frac{Q}{\sqrt[4]{3}}=9.6581974 \qquad \log_{10}\frac{Q}{6\sqrt[4]{3}}=8.8800461$$

The coefficients a_1, a_2, a_3 in each of the four series are given in Table XVII. For practical purposes the formula should be used only for values of $\sqrt{\frac{J-1}{J}}$ smaller than about 0.2.

For the special case $\sqrt{\frac{J-1}{J}}=0$, it is of interest to note that the mutual inductance is given by the simple expression

$$\frac{M}{4\pi}=\left[\frac{x^2}{(x^4+12A^2a^2)^{\frac{1}{4}}}\cdot\frac{P}{\sqrt[4]{27}}-(x^4+12A^2a^2)^{\frac{1}{4}}\cdot\frac{Q}{\sqrt[4]{3}}\right]$$

which, remembering that $x^4=36A^2a^2$ in this case, becomes

$$M=4\pi\sqrt{Aa}(P-2Q)=4\pi(0.112888542)\sqrt{Aa}$$

$$=1.418599262\sqrt{Aa} \qquad [19]$$

If we introduce the distances r_1 and r_2 (Fig. 1) into the formula for $\sqrt{\frac{J-1}{J}}$, we see that the necessary and sufficient condition that this remarkably simple formula [15a] may be used is that $r_1^2=2r_2^2$, or $k'=k=\frac{1}{\sqrt{2}}$. That is, the greatest distance between the two circumference must be $\sqrt{2}$ times the shortest distance between them. The most important cases satisfying this condition are

$\frac{a}{A}$	d	
1	$2A$	Equal circles.
$3-2\sqrt{2}$	0	Circles in the same plane.
$\frac{1}{2}$	$\frac{1}{2}\sqrt{7}\,A$	

[15a] Nagaoka has recently shown (Tokyo Math. Phys. Soc., **6**, p. 10; 1911) that formula (19) may be derived from Maxwell's formula (1).

The convergence of the formula (18) will of course be satisfactory for moderate deviations on the either side of the ideal ratio of $\frac{d}{A}$, but the formula must be regarded as of more limited application than most of those above. It gives, however, a very rapid and accurate means of checking other formulas, since in the ideal case the mutual inductance can be calculated by (19) to any number of decimal places desired, according to the number of figures retained in Stirling's constants P and Q.

CHOICE OF FORMULAS

With so many to choose among, it is possible to select a favorable formula for any individual case. For this purpose r_1 and r_2, the longest and shortest distances between the circles, need to be considered, since on their relative values the convergence or convenience of the various formulas for calculation depends. The following table gives roughly the range of values of the ratio $\frac{r_2}{r_1}$ within which the different formulas are capable of giving the best results. Since, however, the determination of such limits is somewhat arbitrary, the values given here should not be regarded as more than a guide. In the case of those formulas which occur in the form of a series the limiting value of the ratio $\frac{r_2}{r_1}$ has been calculated which makes the last term included not greater than one ten-thousandth of the whole. The values of $\frac{r_2}{r_1}$ for Nagaoka's formulas have been calculated for the limits of his correction tables.

SUMMARY OF FORMULAS FOR CIRCLES

Formula		Range of values of $\frac{r_2}{r_1}$	Most favorable values of $\frac{d}{A}$ for equal circles
Weinstein's	(7)	0 to 0.25	0 to 0.5
Maxwell's	(10)	0 to 0.02	0 to 0.04
"	(12)	0 to 0.14	0 to 0.3
"	(14)	0 to 0.22	0 to 0.45
"	(3)	0 to 0.2	0 to 0.4
"	(2)	0.02 to 0.20	0.04 to 0.4

Formula		Range of values of $\frac{r_2}{r_1}$	Most favorable values of $\frac{d}{A}$ for equal circles
Havelock's	(16)	0 to 0.4	0 to 0.9
Coffin's	(13)		0 to 0.9
Nagaoka's	(9)	0.04 to 0.4	0.08 to 0.9
Maxwell's	(4)	0 to 0.75	0 to 2.25
"	(1)	0.2 to 0.7	0.4 to 2
Mathy	(18)	0.65 to 0.75	1.75 to 2.25
"	(19)	$\frac{1}{2}\sqrt{2}$	2
Nagaoka's	(8)	0.3 to 1	greater than 0.6
Maxwell's	(6)	0.6 to 1	" " 1.5
Havelock's	(17)	0.9 to 1	" " 4
Maxwell's	(5)	0.98 to 1	" " 10

EXAMPLES TO ILLUSTRATE AND TEST THE FORMULAS

EXAMPLE 1. MAXWELL'S FORMULA (1). FOR ANY COAXIAL CIRCLES

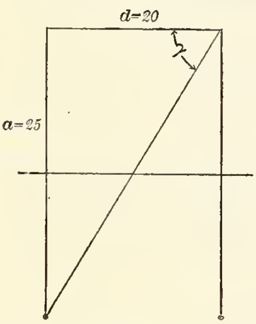

Fig. 3

Let $a = A = 25$ cm, Fig. 3,
$d = 20$ cm.

$$k = \frac{50}{\sqrt{2500+400}} = 0.9284766 = \sin \gamma$$

$$\gamma = 68° \ 11' \ 54''88 = 68°.198578.$$

From Legendre's tables, we obtain

$$\log F = 0.3852191 \qquad \left(\frac{2}{k} - k\right)F - \frac{2}{k}E = 0.5318500$$
$$\log E = 0.0547850$$

$$4a = 100 \qquad \therefore \quad M = 167.08562 \text{ cm.}$$

To facilitate calculations in such problems as this, we have prepared Table II, which gives F and $\log F$, E and $\log E$, as functions of $\tan \gamma$. In the above case $\tan \gamma = \frac{50}{20} = 2.5$, and from Table II we can take the values of $\log F$ and $\log E$ directly, avoiding the calculation of γ and the interpolation for $\log F$ and $\log E$ in Legendre's tables (or Table XIII). This is only applicable for circles of equal radii, and is especially advantageous when $\tan \gamma$ is one of the values given in the table, when interpolation is unnecessary.

The above problem may also be calculated by means of Table I, taken from Maxwell, as follows:

$$\log_{10}\frac{M}{4\pi a} \text{ for } 68^\circ\!.1 \qquad = \bar{1}.7230634$$

$$\text{for } 68^\circ\!.2 \qquad = \bar{1}.7258281$$

$$\text{for } 68^\circ\!.198578 = \bar{1}.7257888 = \log\frac{M}{4\pi a}$$

$\therefore M = 167.08546$ cm, agreeing almost exactly with the above value.

The calculation of mutual inductance by the above methods is simplest for circles not near each other, as then the values of $\log F$, $\log E$, and $\log\frac{M}{4\pi\sqrt{Aa}}$ are very exact when taken by simple interpolation. When γ is nearly 90°, however, second and third differences have to be used in interpolation.

EXAMPLE 2. MAXWELL'S SECOND EXPRESSION (2). FOR CIRCLES NEAR EACH OTHER

Let $a = A = 25$ cm, $d = 1$ cm

In this case $k = \sin\gamma = \frac{50}{\sqrt{2501}} = 0.9998002 \qquad \gamma = 88^\circ\ 51'\ 14''$

This value of γ is so nearly 90° that it is difficult to obtain accurate values of F and E from tables of elliptic integrals, or of $\frac{M}{4\pi a}$ from Maxwell's table.

We may therefore use formula (2) instead of (1).

$$r_1 = \sqrt{2501} = 50.01 \text{ nearly}, \ r_2 = 1.0$$

$$\therefore k_1 = \sin\gamma_1 = \frac{4Aa}{(r_1 + r_2)^2} = 0.9607920$$

$$\gamma_1 = 73^\circ\ 54'\ 9''\!.67 = 73^\circ\!.902687$$

From Legendre's tables or Table XIII, for $\gamma_1 = 73^\circ\!.902687$, $F_1 = 2.7024545$

$E_1 = 1.0852170$

$F_1 - E_1 = 1.6172375$

$$\frac{8\pi\sqrt{Aa}}{\sqrt{k_1}} = \frac{200\pi}{\sqrt{.9607920}} \quad \therefore \frac{8\pi\sqrt{Aa}}{\sqrt{k_1}}\left(F_1 - E_1\right) = M = 1036.6664 \text{ cm.}$$

EXAMPLE 3. FORMULA (3). SERIES FOR F AND E, CIRCLES NEAR EACH OTHER

Suppose that, in the last example, we calculate F and E by means of formula (3), instead of taking them from Table XIII.

$$A = a = 25, \; d = 1.$$

$$k'^2 = 1 - k^2 = 1 - \frac{2500}{2501} = \frac{1}{2501}$$

$$\therefore F = 5.2989471 \quad E = 1.0009594$$

If these values of F and E be substituted in formula (1), k being 0.9998002, we obtain $M = 1036.6652$, which is very closely the same value as by formula (2).

EXAMPLE 4. FORMULA (3). SECOND CASE, CIRCLES NOT NEAR

$$A = 25, \; a = 20, \; d = 10 \text{ cm.} \quad \text{(See Fig. 1.)}$$

$$k^2 = \frac{4 \times 20 \times 25}{(45)^2 + (10)^2} = \frac{16}{17} \quad \therefore k'^2 = \frac{1}{17}$$

$$\log\frac{4}{k'} = \frac{1}{2}\log(16 \times 17) = \frac{1}{2}\log_e 272 = 2.8029010$$

$$\frac{k'^2}{4}\left(\log\frac{4}{k'} - 1\right) = .0265132$$

$$\frac{9k'^4}{64}\left(\log\frac{4}{k'} - \frac{7}{6}\right) = .0007962$$

$$\frac{25k'^6}{256}\left(\log\frac{4}{k'} - \frac{111}{90}\right) = .0000312$$

$$\frac{1225k'^8}{16384}\left(\log\frac{4}{k'} - 1.27\right) = .0000014$$

$$\therefore F = 2.8302430$$

$$1 + \frac{k'^2}{2}\left(\log\frac{4}{k'} - \frac{1}{2}\right) = 1.0677324$$

$$\frac{3k'^4}{16}\left(\log\frac{4}{k'} - \frac{13}{12}\right) = .0011156$$

$$\frac{15k'^6}{128}\left(\log\frac{4}{k'} - 1.20\right) = .0000381$$

$$\frac{175k'^8}{2048}\left(\log\frac{4}{k'} - 1.25\right) = .0000017$$

$$\therefore E = 1.0688878$$

To find the value of M we now use equation (1).

$$\left\{\left(\frac{2}{k}-k\right)F-\frac{2}{k}E\right\}=0.885388$$

Multiplying by $4\pi\sqrt{Aa}=4\pi\sqrt{500}$ gives

$$M=248.7875 \text{ cm.}$$

EXAMPLE 5. FORMULA (4). CIRCLES NEAR TOGETHER

$$A=a=25 \qquad d=4$$

$$k=\frac{2\sqrt{Aa}}{\sqrt{(A+a)^2+d^2}}=\frac{50}{\sqrt{2516}}=0.9968154$$

$$k'^2=\frac{(A-a)^2+d^2}{(A+a)^2+d^2}=\frac{16}{2516}=0.0063593015$$

$$k_0'=\frac{k'^2}{(1+k)^2}=0.0015949004$$

$$\log_e\frac{4}{k_0'}=\log_e 2507.9937=7.8272373$$

$$\frac{k_0'^2}{4}\left(\log_e\frac{4}{k_0'}-1\right)=\frac{0.0000043}{7.8272416}=F_0$$

$$1+\frac{k_0'^2}{2}\left(\log_e\frac{4}{k_0'}-\frac{1}{2}\right)=1.0000093=E_0$$

$$\frac{F_0}{k(1+k)}=3.9323856$$

$$\frac{E_0}{k}(1+k)=\underline{2.0032134}$$

$$1.9291722$$

Multiplying by $4\pi\sqrt{Aa}$ gives

$$M=606.0674 \text{ cm.}$$

If we calculate M by formula (3) we find that, to obtain the same precision, terms in k'^4 in the series for F and E have to be included, and we find

$$M=606.0678 \text{ cm.}$$

EXAMPLE 6. FORMULA (5). CIRCLES FAR APART

$$A = a = 10 \qquad d = 100$$

$$k = \frac{20}{\sqrt{10400}} = \frac{1}{\sqrt{26}} = 0.19611615$$

$$1 + \frac{3}{4}k^2 = 1.02884616$$

$$\frac{75}{128}k^4 = 0.00086684$$

$$\frac{245}{512}k^6 = 0.00002723$$

$$\frac{6615}{128^2}k^8 = \underline{0.00000088}$$

$$\text{Sum} = 1.02974111$$

$$\log \text{sum} = 0.0127281$$

$$\log\ k^3 = \bar{3}.8775400$$

$$\log \frac{\pi^2\sqrt{Aa}}{4} = \underline{1.3922398}$$

$$\log M = \bar{1}.2825079$$

$$M = 0.19164962 \text{ cm.}$$

If formula (1) be used, and the values of F and E be taken by interpolation from Table XII, the value $M = 0.191643$ is found, which is in error by more than 5 parts in 10000. Using the formula (6) terms in k_1^2 only need be calculated, and we find $M = 0.19164958$, which differs by only one part in five million from the value given by (5).

EXAMPLE 7. FORMULA (6). CIRCLES NOT NEAR TOGETHER

$$A = 25 \qquad a = 20 \qquad d = 40$$

$$r_1 = \sqrt{3625} \qquad r_2 = \sqrt{1625}$$

$$k_1 = \frac{4Aa}{(r_1 + r_2)^2} = 0.19793905$$

$$1+\frac{3}{8}k_1^2=1.01469245 \qquad \log \text{sum}=0.0064929$$

$$\frac{15}{64}k_1^4=0.00035978 \qquad \log k_1^{\frac{3}{2}}=\bar{2}.9447972$$

$$\frac{175}{1024}k_1^6=0.00001028 \qquad \log \frac{\pi}{2}\sqrt{Aa}=\underline{1.5456049}$$

$$\frac{1225}{128^2}k_1^8=\underline{0.00000032} \qquad \log \frac{M}{4\pi}=0.4968950$$

$$\text{Sum}=1.0150628$$

$$\therefore\ \frac{M}{4\pi}=3.1397496 \text{ cm}$$

By formula (8) $\frac{M}{4\pi}=3.1397486$

If formula (1) is used and the elliptic integrals be taken from Table XII by interpolation the value $\frac{M}{4\pi}=3.1397656$ is found, which is only five in a million in error.

EXAMPLE 8. WEINSTEIN'S FORMULA (7). FOR ANY COAXIAL CIRCLES NOT TOO FAR APART

Take the same circles as in example 4.

$$A=25,\ a=20,\ c=5,\ d=10$$

$$k'^2=\frac{1}{17},\ \log\frac{4}{k'}-1=1.802901$$

$$1+\frac{3}{4}k'^2=1.0441176 \qquad 1+\frac{15}{128}k'^4=\ \ 1.0004053$$

$$\frac{33}{64}k'^4=\ .0017842 \qquad \frac{185}{1536}k'^6=\ .0000245$$

$$\frac{107}{256}k'^6=\ .0000851 \qquad \frac{7465}{65536}k'^8=\ \underline{.0000012}$$

$$\frac{5913}{16384}k'^8=\ \underline{.0000042} \qquad 1.0004310=C$$

$$\text{Sum}\ =1.0459911=B$$

$$B\log\left(\frac{4}{k'}-1\right)=1.8858184;\ \left\{B\log\left(\frac{4}{k'}-1\right)-C\right\}=0.8853874$$

Multiplying by $4\pi\sqrt{500}$ gives $M=248.7873$ cm, agreeing almost exactly with the value previously found, example 4.

EXAMPLE 9. NAGAOKA'S FORMULA (8). CIRCLES NOT NEAR TOGETHER

$$A=a=25 \qquad d=20 \qquad \text{(See Fig. 3.)}$$

$$\sqrt{k'}=\left(\frac{20}{\sqrt{2900}}\right)^{\frac{1}{2}}=0.6094183$$

$$\frac{l}{2}=\frac{1-\sqrt{k'}}{2(1+\sqrt{k'})}=\frac{1}{2}\,\frac{0.3905817}{1.6094183}=0.12134250$$

$$\text{From Table XV, } q-\frac{l}{2} = 0.00005269$$

$$\therefore q = 0.12139519$$

$$\frac{3}{2}\log q = \bar{2}.6263022$$

$$\text{From Table XV, } \log(1+\epsilon) = 0.0002775$$

$$\log 400\pi^2 = 3.5963598$$

$$\log M = 2.2229395$$

$$\therefore M = 167.08577 \text{ cm}$$

or about one in a million higher than the value found for the same circles in example 1.

EXAMPLE 10. NAGAOKA'S FORMULA (8). CIRCLES FAR APART

$$A=a=10 \qquad d=100$$

$$k'=\frac{100}{\sqrt{10400}}=0.98058073$$

$$k=\frac{20}{\sqrt{10400}}=0.19611615$$

$$1+\sqrt{k'}=1.9902427$$

$$\frac{l}{2}=\frac{1}{2}\,\frac{k^2}{(1+k')(1+\sqrt{k'})^2}=0.0024512756$$

The differences $q-\frac{l}{2}$ and ϵ are negligible, so that we have

$$M=16\pi^2\sqrt{Aa}\left(\frac{l}{2}\right)^{\frac{3}{2}}=0.19164966 \text{ cm}$$

which is in very close agreement with the valves found by formulas (5) and (6) and Havelock's formula (17) for the same pair of circles. If we calculate $\frac{l}{2}$ by the formula $\frac{1}{2}\frac{(1-\sqrt{k'})}{(1+\sqrt{k'})}$ instead we find difficulty in obtaining $(1-\sqrt{k'})$ with sufficient precision. The value of M found by using this formula for $\frac{l}{2}$ and with seven place logarithms is in this case $M=0.19164980$, or about one part in a million different.

EXAMPLE 11. NAGAOKA'S SECOND FORMULA (9). FOR CIRCLES NEAR EACH OTHER

$$A=a=25 \qquad d=4$$

$$k=\sin\gamma=\frac{50}{\sqrt{2516}}=0.99681535 \qquad \sqrt{k}=0.99840640$$

$$k'^2=\frac{\sqrt{(A-a)^2+d^2}}{\sqrt{(A+a)^2+d^2}}=\frac{4}{\sqrt{2516}}=0.0063593014$$

$$\frac{l_1}{2}=\frac{k'^2}{(1+k)(1+\sqrt{k})^2}=0.00039872542=q_1$$

as $\left(\frac{l_1}{2}\right)^5$ and higher powers can be neglected.

$$\log_e\left(\frac{1}{q_1}\right)=\log_e 2507.9919=7.8272376$$

From Table XVI $\qquad -\epsilon_1'=0.00000128$

$$8q_1+\epsilon_1'=0.00318852$$

$$[1+8q_1+\epsilon_1']\log_e\left(\frac{1}{q_1}\right)-4=3.8521929 \quad =P$$

$$\frac{1}{2(1-2q_1)^2}=0.50079850=Q$$

$$\therefore\ M=4\pi\sqrt{Aa}\cdot PQ=606.0674\text{ cm}$$

which is exactly the same value as was found for the same circles in example 5.

Using Table II for the above problem, where $\tan\gamma=12.5$, we have $\log F=0.5932708$ and $\log E=0.0047004$. Using these values in formula (1) we obtain for the mutual inductance

$$M=606.0666 \text{ cm}$$

which differs from the value by Nagaoka's formula by 1 part in a million.

EXAMPLE 12. MAXWELL'S SERIES FORMULA (10). FOR ANY TWO COAXIAL CIRCLES NEAR EACH OTHER

$$A=26 \qquad a=25 \qquad d=1 \qquad c=1 \qquad r=\sqrt{2}$$

$$\text{Since } r=\sqrt{2}, \qquad \log_e \frac{8a}{r}=\log_e \frac{200}{\sqrt{2}}=4.9517438$$

$$\begin{array}{rrrr}
1+\dfrac{c}{2a}= & 1.0200000 & 2+\dfrac{c}{2a}= & 2.0200000 \\
\dfrac{c^2+3d^2}{16a^2}= & .0004000 & -\dfrac{3c^2-d^2}{16a^2}= & -\ .0002000 \\
-\dfrac{c^3+3cd^2}{32a^3}= & -\ .0000080 & +\dfrac{c^3-6cd^2}{48a^3}= & -\ .0000067 \\
 & \overline{1.0203920}=B & & \overline{2.0197933}=C
\end{array}$$

$$\begin{array}{rl}
B\log\dfrac{8a}{r}= & 5.0527192 \\
C\ = & \underline{2.0197933} \\
\left\{B\log\dfrac{8a}{r}-C\right\}= & 3.0329259 \text{ Multiply by } 4\pi a=100\pi \text{ and}
\end{array}$$

$$M=952.8218 \text{ cm.}$$

This formula would be less accurate for the circles of problem 4, but is accurate for circles close together, as this problem shows.

EXAMPLE 13. MAXWELL'S FORMULA (12). FOR CIRCLES OF EQUAL RADII NEAR EACH OTHER

$$A=a=25 \qquad d=1$$

$$\frac{8a}{d}=200 \qquad \log_e 200=5.298317$$

$$\log_e \frac{8a}{d}\left(1+\frac{3d^2}{16a^2}\right)=1.000300\times 5.298317=5.29990$$

$$\left(2+\frac{d^2}{16a^2}\right)=\frac{2.00010}{3.29980}$$

Multiply by $4\pi a=100\pi$

$$M=1036.663 \text{ cm}$$

nearly agreeing with the more exact value found under problem 2.

This is a very simple and convenient formula for equal circles and gives approximate results for circles still farther apart than in this problem.

EXAMPLE 14. HAVELOCK'S FORMULA (16). FOR CIRCLES AT MODERATE DISTANCES

$$A=25 \qquad a=20 \qquad d=10 \qquad \therefore c=5$$

$$r=5\sqrt{5} \qquad \alpha=\frac{r^2}{Aa}=\frac{1}{4} \qquad \frac{8\sqrt{Aa}}{r}=16$$

$$\begin{aligned}
\log_e 16 &= 2.7725887\\
1+\frac{3}{16}\alpha &= 1.0468750\\
-\frac{15}{1024}\alpha^2 &= -0.0009155\\
\frac{35}{128^2}\alpha^3 &= 0.0000334\\
-\frac{1575}{2.128^3}\alpha^4 &= -0.0000015\\
\text{Sum} &= 1.0459914\\
\text{Multiplied by } \log_e 16 &= 2.9001037=B
\end{aligned}$$

$$\begin{aligned}
2+\frac{1}{16}\alpha &= 2.0156250\\
-\frac{31}{2048}\alpha^2 &= -0.0009461\\
\frac{247}{6.128^2}\alpha^3 &= 0.0000393\\
-\frac{7795}{8.128^3}\alpha^4 &= -0.0000018\\
\text{Sum} &= 2.0147164=C\\
B-C &= 0.8853873\\
\text{Multiplied by } 4\pi\sqrt{Aa} &= 248.7873 \text{ cm}=M
\end{aligned}$$

which agrees exactly with the value found in example 8.

If the example 12 be calculated by this formula, no terms of order higher than α^2 need be calculated, and

$$
\begin{aligned}
& M = 952.8221 \text{ cm} \\
\text{Formula (10)}\ & M = 952.8218 \\
\text{Formula (3)}\ & M = 952.8219
\end{aligned}
$$

EXAMPLE 15. COFFIN'S FORMULA (13). EXTENSION OF FORMULA (12) FOR CIRCLES OF EQUAL RADII

$$A = a = 25 \qquad d = 16$$

$$\frac{8a}{d} = 12.5 \qquad \log_e 12.5 = 2.5257286$$

$$\text{First series of terms} = B = 1.074478$$

$$\text{Second series of terms} = C = 2.023220$$

$$\therefore \left\{ B \log \frac{8a}{d} - C \right\} = 0.690620$$

$$4\pi a = 100\pi \quad \therefore M = 216.9647 \text{ cm.}$$

This agrees with the value given by formula (1) within 1 part in 200,000. As the distance apart of the circles increases the accuracy by this formula of course gradually decreases.

EXAMPLE 16. FORMULA (14). EXTENSION OF MAXWELL'S FORMULA (10) FOR CIRCLES OF UNEQUAL RADII

$$A = 25 \qquad a = 20 \qquad c = 5 \qquad d = 10$$

$$r = \sqrt{c^2 + d^2} = 5\sqrt{5} \qquad \log_e \frac{8a}{r} = \log_e \frac{32}{\sqrt{5}} = 2.6610169$$

$$
\begin{aligned}
\text{First series of terms} = B \log_e \frac{8a}{r} &= 3.112060 \\
\text{Second series of terms} = C \quad &= \underline{2.122114} \\
&\ 0.989946 \\
\text{multiplying by } 4\pi a = 80\pi \quad & M = 248.8006 \text{ cm.}
\end{aligned}
$$

This result is correct to 1 part in 19,000 (see examples 4, 8, and 14). Using only the first three terms for B and C (that is, formula 10), the result would be too large by 1 part in 1750.

EXAMPLE 17. HAVELOCK'S FORMULA (17). CIRCLES FAR APART

$$a = 10 = A \qquad d = 100$$

$$\frac{a}{A} = 1 \qquad \frac{A}{d} = 0.1$$

$$1-\frac{3}{2}\cdot 2\cdot\frac{A^2}{d^2}=\quad 0.9700000$$

$$\frac{15}{8}\cdot 5\cdot\frac{A^4}{d^4}=\quad 0.0009375$$

$$-\frac{35}{16}\cdot 14\cdot\frac{A^6}{d^6}=-0.0000306$$

$$\frac{315}{128}\cdot 42\cdot\frac{A^8}{d^8}=\quad \underline{0.0000010}$$

$$\text{Sum}=\quad 0.9709079$$

$$\text{Multiplied by }\frac{2\pi^2A^2a^2}{d^3}=\quad 0.19164958\text{ cm}=M$$

which is in exact agreement with the value found by formula (6).

EXAMPLE 18. MATHY'S FORMULA (18)

$$A=25\qquad a=20\qquad d=40$$

$$x^2=625+400+1600=2625$$

$$\frac{Aa}{x^2}=\frac{500}{5625}=\frac{4}{21}$$

$$x^4+12\,A^2a^2=9890625$$

$$\frac{1}{4}\log\,(x^4+12A^2a^2)=1.7488059$$

$$\log\left[\frac{x^2}{(x^4+12A^2a^2)^{\frac{1}{4}}}\right]=1.6703234$$

$$1-36\,\frac{A^2a^2}{x^4}=-0.3061225$$

$$1+12\,\frac{A^2a^2}{x^4}=\quad 1.4353742$$

$$\sqrt{\frac{J-1}{J}}=-0.17801131$$

$$\log z=\quad \bar{2}.5008952$$

Using the constants in Table XVII we calculate the four series

$F\left(\frac{1}{12},\frac{5}{12},\frac{1}{2},z\right)$	$F\left(\frac{7}{12},\frac{11}{12},\frac{3}{2},z\right)$	$F\left(-\frac{1}{12},\frac{7}{12},\frac{1}{2},z\right)$	$F\left(\frac{5}{12},\frac{13}{12},\frac{3}{2},z\right)$
1.0022006	1.0112962	0.9969192	1.0095357
0.0000357	0.0002173	−0.0000473	0.0001784
0.0000008	0.0000049	−0.0000010	0.0000040
1.0022371	1.0115184	0.9968709	1.0097181

From these, using the values of the constants already calculated, we find the four terms in the formula for M

$$\begin{array}{rr} C= & 26.981438 \\ D= & -\ 0.639427 \\ \hline C-D= & 27.620865 \end{array} \qquad \begin{array}{rr} G= & 25.447327 \\ H= & -\ 0.966215 \\ \hline G-H= & 24.481112 \end{array}$$

$$\frac{M}{4\pi}=27.620865-24.481112=3.139753$$

By Nagaoka's formula (8) we find $\frac{M}{4\pi}=3.1397496$

" " (6) " " $\frac{M}{4\pi}=3.1397486$

Mathy's formula suffers here under the inconvenience that M is given as the difference of two quantities considerably larger than itself.

EXAMPLE 19. FORMULA (19). FOR CIRCLES SATISFYING THE CONDITION $r_1^2=2r_2^2$ OR $k'=k=\frac{1}{\sqrt{2}}$

$$A=a=25 \qquad d=50$$

$$\begin{aligned} M &= 1.41859262 \cdot\cdot\cdot\cdot \sqrt{Aa} \\ &= 35.4649816 \cdot\cdot\cdot\cdot \text{ cm.} \end{aligned}$$

By Nagaoka's formula (8), $M=35.464975$
" " (6), $M=35.464981$
" " (1), $M=35.46481$

We see that the formulas (8) and (6) here give an accuracy limited only by that of the logarithm tables. The result found by formula (1), using Table XIII, is, however, affected by the fact that the value of the quantity in the parentheses $(1.4239167-1.3110287)$, is only about a tenth as large as the numbers of which it is the difference.

2. MUTUAL INDUCTANCE OF TWO COAXIAL COILS

ROWLAND'S FORMULA

Let there be two coaxial coils of mean radii A and a, axial breadth of coils b_1 and b_2, radial depth c_1 and c_2, and distance apart of their mean planes d. Suppose them uniformly wound with n_1 and n_2 turns of wire. The mutual inductance M_0 of the two central turns of the coils (Fig. 4), will be given by formula (1) or (7), or any one of the foregoing formulas for the mutual inductance of coaxial circles adapted to the particular case may be used, and the mutual inductance M of the two coils of n_1 and n_2 turns will then be, to a *first approximation*,

$$M = n_1 n_2 M_0$$

The following second approximation was obtained by Rowland by means of Taylor's theorem, following Maxwell, § 700:

$$\frac{M}{n_1 n_2} = M_0 + \frac{1}{24}\left\{(b_1^2 + b_2^2)\frac{d^2M_0}{dx^2} + c_1^2\frac{d^2M_0}{da^2} + c_2^2\frac{d^2M_0}{dA^2}\right\}$$

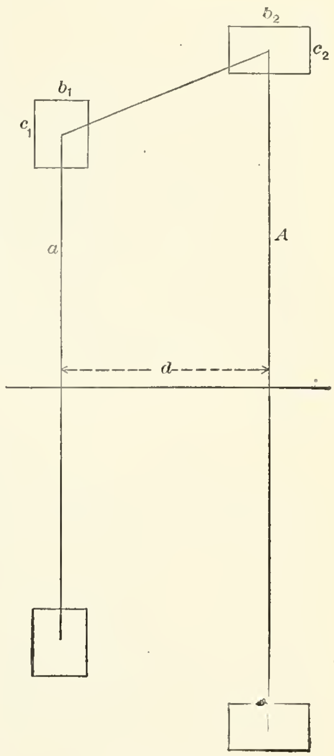

Fig. 4

If the two coils are of equal radii but unequal section,

$$\frac{M}{n_1 n_2} = M_0 + \frac{1}{24}\left\{(b_1^2 + b_2^2)\frac{d^2M_0}{dx^2} + (c_1^2 + c_2^2)\frac{d^2M_0}{da^2}\right\} \qquad [20]$$

If the two coils are of equal radii and equal section, this becomes

$$\frac{M}{n_1 n_2} = M_0 + \frac{1}{12}\left\{b^2\frac{d^2M_0}{dx^2} + c^2\frac{d^2M_0}{da^2}\right\} \qquad [21]$$

The correction terms will be calculated by means of the following:

$$\begin{aligned} \frac{d^2M_0}{da^2} &= \pi\frac{k}{a}\left\{(2-k^2)\,F - \left(2-k^2\frac{1-2k^2}{1-k^2}\right)E\right\} \\ \frac{d^2M_0}{dx^2} &= \pi\frac{k^3}{a}\left\{F - \frac{1-2k^2}{1-k^2}E\right\} \end{aligned} \qquad [22]$$

The equation (21) is equivalent to Rowland's equation, where 2ξ and 2η are the breadth and depth of the section of the coil, instead of b and c, except that there is an error in the formula as printed in Rowland's [16] paper, ξ and η being interchanged. The equations (22) are equivalent to those given by Rowland, being somewhat simpler.[17]

Formula (21) gives a very exact value for the mutual inductance of two coils, provided the cross sections are relatively small and the distance apart d is not too small. But when b or c is large or d is small the fourth differential coefficients which have been neglected become appreciable and the expression may not be sufficiently exact.

RAYLEIGH'S FORMULA

Maxwell [18] gives a formula, suggested by Rayleigh, for the mutual inductance of two coils, which has a very different form from Rowland's, but is nearly equivalent to it when the coils are not near each other. It has been used by Rayleigh in calculating the mutual inductance of a Lorenz apparatus and by Glazebrook (Phil. Trans., 1883) in calculating the mutual inductance of parallel coils of rectangular section employed in a determination of the ohm. It may also be employed in calculating the attraction between two coils.[19] It is sometimes called the formula of quadratures, and is as follows: [20]

$$M = \frac{1}{6}\left(M_1 + M_2 + M_3 + M_4 + M_5 + M_6 + M_7 + M_8 - 2M_0\right) \qquad [23]$$

[16] Collected Papers, p. 162. Am. Jour. Sci. [3], XV, 1878.

[17] Gray, Absolute Measurements, Vol. II, Part II, p. 322.

[18] Electricity and Magnetism, Vol. II, Appendix II, Chapter XIV.

[19] Gray, Absolute Measurements, Vol. II, Part II, p. 403.

[20] This Bulletin, 2, p. 370–372; 1906.

where M_1 is the mutual inductance of the circle O_2 and a circle through the point 1 of radius $A-\frac{c_1}{2}$, and similarly for the others, Fig. 5.

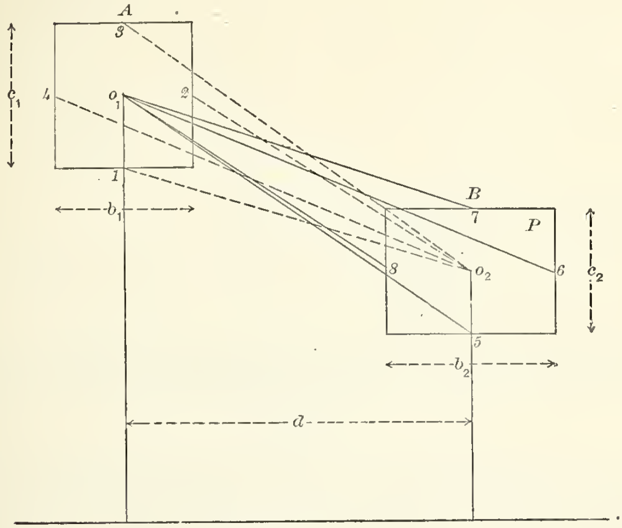

Fig. 5

For two coils of equal radii and equal section this becomes

$$M=\frac{1}{3}\left(M_1+M_2+M_3+M_4-M_0\right) \qquad [24]$$

Equation (23) is Rayleigh's formula, or the formula of quadratures. Instead of computing the correction to M_0 by means of the differential coefficients (20), eight additional values are computed, corresponding to the mutual inductances of the single turns at the eight numbered points indicated in Fig. 5, each with reference to the central turn of the other coil. These M's may be computed by any of the formulas for the mutual inductance of coaxial circles which may be best adapted to the particular case, and the values of the constants for the case of two coils of *equal radii* are given in the following table, the radius being a in every case.

	Axial distance	Radial distance	r
Using (10)	d	$-\frac{c_1}{2}$	$\sqrt{d^2+\frac{c_1^2}{4}}$
"	d	$+\frac{c_1}{2}$	$\sqrt{d^2+\frac{c_1^2}{4}}$
"	d	$-\frac{c_2}{2}$	$\sqrt{d^2+\frac{c_2^2}{4}}$
"	d	$+\frac{c_2}{2}$	$\sqrt{d^2+\frac{c_2^2}{4}}$
Using (12)	$d-b_1/2$	0	
"	$d+b_1/2$	0	
"	$d+b_2/2$	0	
"	$d-b_2/2$	0	

MAGNITUDE OF THE ERRORS IN ROWLAND'S AND RAYLEIGH'S FORMULAS

The error ϵ_1 in equation (24), for two coils of equal radii a, distance between centers being d, and section $b \times c$ (Fig. 6), depends on the dimensions of the coil in a manner shown by the following expression:[21]

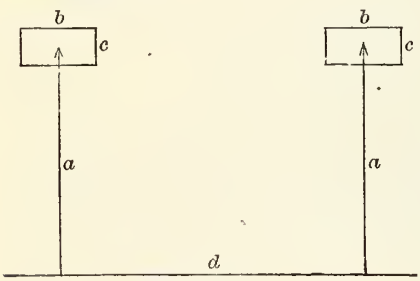

Fig. 6

$$\epsilon_1 \propto 4\pi a \left\{ \frac{3b^4+3c^4-20b^2c^2}{480d^4} \right\} \qquad [25]$$

For a square coil the correction is a negative quantity, showing that M by equation (24) is too large, and the error is proportional to the fourth power of $\frac{1}{d}$, the reciprocal of the distance between the mean planes of the coils. For a rectangular coil in which b is greater than c the correction is negative so long as b is not more than 2.5 times c. When b is still larger with respect to c the correction becomes plus, the value of M by (24) being too small.

[21] This Bulletin, 2, p. 373; 1906.

Thus, for a coil of cross section 4 sq. cm, we get the following values of the numerator of (25) as we vary the shape of cross section keeping $bc=4$.

Dimensions of coil		Error proportional to—
$b=2$	$c=2$	$-\ 224$
$b=2.5$	$c=1.6$	$-\ 183$
$b=3$	$c=1.33$	$-\ 67.5$
$b=4$	$c=1$	$+\ 451$
$b=8$	$c=0.5$	$+11,988$

Thus we see that the value of M as given by the formula of quadratures may be too large or too small according to the shape of the section, and that the error is proportional directly to the fourth power of the dimensions of the section and inversely to the fourth power of the distance between the mean planes of the coils. When the section is small and d large the error will become negligible.

The error by Rowland's formula is—[22]

$$\epsilon_2 \propto 4\pi a \frac{6}{d^4}\left\{\frac{b^4+c^4}{360}-\frac{b^2c^2}{144}\right\} \propto 4\pi a\left\{\frac{8b^4+8c^4-20b^2c^2}{480d^4}\right\} \qquad [26]$$

This is negative for a square coil, but smaller than ϵ_1. For a coil of section such that $b=c\sqrt{2}$, the error is zero, and for sections such that $\frac{b}{c}>\sqrt{2}$, the error is positive. Thus, for a coil of cross section 4 sq. cm, we get the following values of the numerator of (26) which is proportional to the error by Rowland's formula.

Dimensions of coil		Error proportional to—
$b=2$	$c=2$	$-\ 64$
$b=2.5$	$c=1.6$	$+\ 45$
$b=3$	$c=1.33$	$+\ 353$
$b=4$	$c=1$	$+\ 1,736$
$b=8$	$c=0.5$	$+32,448$

Thus the error is smaller by Rowland's formula for coils having square or nearly square section, but larger for coils having rectangular sections not nearly square.

[22] This Bulletin, **2**, p. 373; 1906.

LYLE'S FORMULA

Professor Lyle [23] has recently proposed a very convenient method for calculating the mutual inductance of coaxial coils, which gives very accurate results for coils at some distance from each other.

The mutual inductance is calculated from formula (1) or any other formula for two coaxial circles, using, however, a modified radius r instead of the mean radius a, r being given by the following equation when the section is square, b being the side of the square section:

$$r = a\left(1 + \frac{b^2}{24a^2}\right) \qquad [27]$$

If the coil has a rectangular section not square, it can be replaced by two filaments (Fig. 7) the distance apart of the filaments being called the *equivalent breadth* or the *equivalent depth* of the coil.

$$\left.\begin{aligned} \beta^2 &= \frac{b^2 - c^2}{12}, \ 2\,\beta \text{ is the equivalent breadth of A} \\ \delta^2 &= \frac{c^2 - b^2}{12}, \ 2\cdot\delta \text{ is the equivalent depth of B} \end{aligned}\right\} \qquad [28]$$

The equivalent radius of A is given by the same expression which holds for a square coil, viz:

$$r = a\left(1 + \frac{c^2}{24a^2}\right)$$

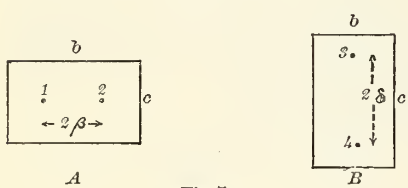

Fig 7.

In the coil B the equivalent filaments have radii $r + \delta$ and $r - \delta$, respectively, where

$$r = a\left(1 + \frac{b^2}{24a^2}\right)$$

The mutual inductance of two coils may now be readily calculated. If each has a square section, it is necessary only to calculate the mutual inductance of the two equivalent filaments. For coils of rectangular sections, as A, B, the mutual inductance will be the sum of the mutual inductances of the two filaments of A on the two

[23] Phil. Mag., 3, p. 310; 1902. Also this Bulletin, 2, pp. 374–378; 1906.

filaments of B, counting $n/2$ turns in each. Or, it is $n_1 n_2$ times the mean of the four inductances M_{13}, M_{14}, M_{23}, M_{24}, where M_{13} is the mutual inductance of filament 1 on filament 3, etc.

Lyle's method is of special value in computing mutual inductances because it applies to coils of unequal as well as of equal radii.

ROSA'S FORMULA[24]

Writing the mutual inductance of two coaxial coils of equal radii and equal section as $\frac{M}{n_1 n_2} = M_0 + \Delta M$, where M_0 is the mutual inductance of the central circles of the two equal coils of sections $b \times c$, Fig. 5, and ΔM is the correction for the section of the coil, the value of ΔM is as follows:

$$\Delta M = 4\pi a \left\{ \frac{3b^2+c^2}{96a^2} \cdot \log\frac{8a}{d} - \frac{11b^2-3c^2}{192a^2} + \frac{b^2-c^2}{12d^2} + \frac{2b^4+2c^4-5b^2c^2}{120d^4} \right.$$
$$+ \frac{6b^4+6c^4+5b^2c^2}{5760a^2d^2} + \frac{3b^6-3c^6+14b^2c^4-14b^4c^2}{504d^6} + \frac{7c^2d^2}{1024a^4}\left(\log\frac{8a}{d} - \frac{163}{84}\right)$$
$$\left. - \frac{15b^2d^2}{1024a^4}\left(\log\frac{8a}{d} - \frac{97}{60}\right) \right\} \qquad [29]$$

For a square section, when $b = c$, this becomes

$$\Delta M = \frac{\pi b^2}{6a}\left\{ \log\frac{8a}{d} - 1 - \frac{a^2b^2}{5d^4} - \frac{3d^2}{16a^2}\left(\log\frac{8a}{d} - \frac{4}{3}\right) + \frac{17b^2}{240d^2} \right\} \qquad [30]$$

The last two terms of equation (30) are relatively small, so that we may write, *approximately:*

$$\Delta M = \frac{\pi b^2}{6a}\left\{ \log\frac{8a}{d} - 1 - \frac{a^2b^2}{5d^4} \right\} \qquad [31]$$

For coils of equal radii but unequal sections, the formula is, neglecting differentials of sixth order

$$\Delta M = 4\pi a \left\{ \frac{3(b_1^2+b_2^2)+(c_1^2+c_2^2)}{192a^2} \log\frac{8a}{d} - \frac{11(b_1^2+b_2^2)-3(c_1^2+c_2^2)}{384a^2} \right.$$
$$+ \frac{(b_1^2+b_2^2)-(c_1^2+c_2^2)}{24d^2} \qquad [32]$$
$$\left. + \frac{(3b_1^4+10b_1^2b_2^2+3b_2^4)+(3c_1^4+10c_1^2c_2^2+3c_2^4)-10(b_1^2+b_2^2)(c_1^2+c_2^2)}{960d^4} \right\}$$

[24] This Bulletin, 4, p. 348, equations (38) and (39).

These expressions for ΔM are very exact where the coils are near together or even where they are separated by a considerable distance, but become less exact as d is greater. They are therefore most reliable where formulas (21), (24), and (27) are least reliable. As formula (31) is exact enough for most purposes, it affords a very easy method of getting the correction for equal coils of square section.

Stefan's formula for the mutual inductance of two equal coaxial coils (originally published[25] without demonstration) is incorrect and is not given here. It resembles equation (29), but is seriously in error for coils at considerable distances.

THE ROSA-WEINSTEIN FORMULA

Weinstein's formula[26] for the mutual inductance of equal coaxial coils has been revised and corrected by Rosa, and the value of ΔM, the correction for section, expressed separately. The expression for ΔM is as follows:[26a]

$$\Delta M = 4\pi a \sin\gamma\left\{(F-E)\left(A+\frac{c^2}{24a^2}\right)+EB\right\} \qquad [33]$$

where F and E are the complete elliptic integrals to modulus $\sin\gamma$, Fig. 8 (as in equation 1),

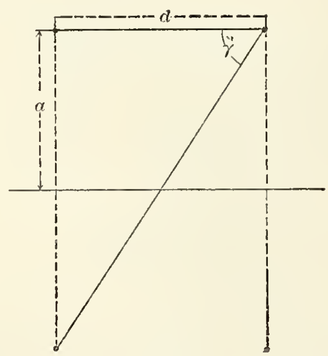

Fig. 8

[25] Wied. Annalen, 22, p. 107; 1884.
[26] Wied. Annalen, 21, p. 350; 1884.
[26a] This Bulletin, 4, p. 342, equation (20); 1907.

and

$$A=\frac{\cos^2\gamma}{12d^2}\left(\alpha_1-\alpha_2-\alpha_3+(2\alpha_2-3\alpha_3)\cos^2\gamma+8\alpha_3\cos^4\gamma\right)$$

$$B=\frac{\sin^2\gamma}{12d^2}\left(\alpha_1+\frac{\alpha_2}{2}+2\alpha_3+(2\alpha_2+3\alpha_3)\cos^2\gamma+8\alpha_3\cos^4\gamma\right)$$

The values of α_1, α_2, and α_3 are as follows:

$$\alpha_1=b^2-c^2+\frac{c^4}{30a^2} \qquad \text{For square section: } \alpha_1=\frac{b^4}{30a^2}$$

$$\alpha_2=\frac{5b^2c^2-4c^4}{60a^2} \qquad \text{“ “ “} \quad \alpha_2=\frac{b^4}{60a^2}$$

$$\alpha_3=\frac{2b^4+2c^4-5b^2c^2}{20d^2} \qquad \text{“ “ “} \quad \alpha_3=-\frac{b^4}{20d^2}$$

Formula (33) is a very exact formula for all positions of the two coils, except when they are very close together.

Weinstein's original formula,[27] which is much less accurate than (33) for coils relatively near together, is not here given.

USE OF FORMULAS FOR SELF-INDUCTANCE IN CALCULATING MUTUAL INDUCTANCE

One can sometimes obtain the mutual inductance of adjacent coils, or of coils at a distance from one another, by means of a formula for the self-inductance of coils. Thus, suppose we have a coil of rectangular section, which we subdivide into three equal parts, 1, 2, 3, Fig. 9. Let L be the self-inductance of the whole coil, L_1 be the self-inductance of any one of the three equal smaller coils, and L_2 be the self-inductance of two adjacent coils taken together. Also let M_{12} be the mutual inductance of coil 1 on coil 2, or of coil 2 on coil 3, and M_{13} be the mutual inductance of coil 1 on coil 3. Then,

$$L=3L_1+4M_{12}+2M_{13}$$

Also, $L_2=2L_1+2M_{12}$

$$\therefore\ M_{12}=\frac{L_2-2L_1}{2} \qquad [34]$$

and $M_{13}=\dfrac{L+L_1-2L_2}{2}$

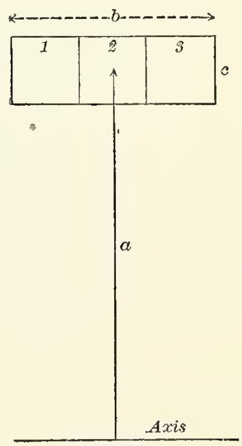

Fig. 9

[27] Wied. Annalen, **21**, p. 350; 1884.

Formula (34) will thus enable us to find the mutual inductance of two coils of equal radii adjacent or near each other by the calculation of self-inductances from such formulas as those of Weinstein (88) and Stefan (90). These latter formulas are not, however, exact enough when the section is large to permit us to apply them to coils at any considerable distance from one another.

GEOMETRIC MEAN DISTANCE FORMULA

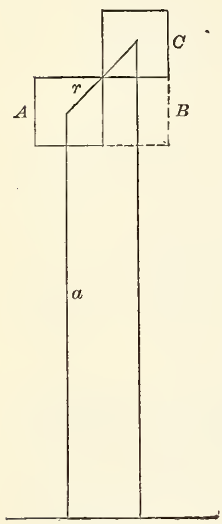

Fig. 10

The mutual inductance of two coaxial coils adjacent or very near can sometimes be obtained by means of the geometric mean distances. This method is accurate only when the sections are very small relatively to the radius. It can often be used to advantage in testing other formulas, but not often in determining the mutual inductance of actual coils.

Formula (10) gives the mutual inductance of two very near coaxial coils in terms of the geometric mean distance, if r be replaced by R, the geometric mean distance of the two sections. Formula (10) gives M_0 if r be used, where r is the distance between centers. Thus,

$$\Delta M = 4\pi a\left(1+\frac{c}{2a}\right)\log\frac{r}{R} \qquad [35]$$

For coils A and C (Fig. 10), $R<r$ and ΔM is positive; $R=0.99770\,r$
" " A " B, $R>r$ and ΔM is negative; $R=1.00655\,r$

The same formula may also be used for squares not adjacent, but only when quite near.[28]

For illustrations and tests of the above formulas, see examples 20–33, pages 44–52.

[28] For other values of the geometric mean distances of squares in a plane see this Bulletin, 3, p. 1; 1907.

CHOICE OF FORMULAS

(*a*) For coils of equal radii and equal cross section (29) should be used if the coils are rather near together. If the cross section is square (29) takes the more simple form (30), and in some cases this may be used in its abbreviated form (31). For coils at all distances, except near together, (33) gives very good precision; (24) and (21) are not so accurate as this last, but give good results if the coils are far apart and their cross sections are not too large.

(*b*) For coils of equal radii but unequal section (32) is accurate for coils not too far away from one another. For coils farther separated (20), (23) or (28) may be used.

(*c*) For coils of unequal radii (23), (24), (27), and (28) apply, but unfortunately they are not as accurate as some of the others. except when the coils are relatively distant or have very small cross sections. The difficulty can be overcome by subdividing each of the two coils into two, four, or more equal parts, and taking the sum of the mutual inductances of all of the parts of one on all the parts of the other. This is a laborious operation, but in important cases it should be done. As the subdivision is carried further the results will approach a final value, and hence the results themselves show when the subdivision has been carried far enough.

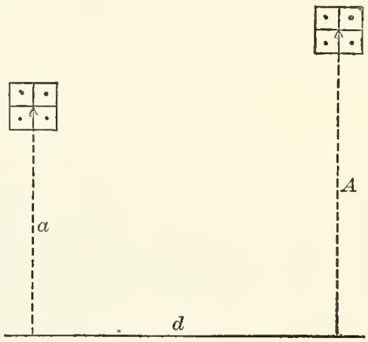

Fig. 11

Thus, suppose two coils A, B (Fig. 11) of square section are subdivided into four equal parts and by the method of Lyle, formula (27), the mutual inductance of the whole of B is computed on each of the four parts of A. If the sum differs appreciably from the result obtained by taking A and B as wholes in one calculation, then the four parts of B may be taken separately with respect to the separate parts of A. If one is doubtful whether this is sufficiently accurate, one of the sections of A may be subdivided further and calculated with respect to one section of B, to see whether there is any appreciable

difference due to this further subdivision. For coils of equal radii very accurate results for near coils can be obtained much more easily by using some of the other formulas.

EXAMPLES TO ILLUSTRATE THE FORMULAS FOR THE MUTUAL INDUCTANCE OF COILS OF RECTANGULAR SECTION

EXAMPLE 20. ROWLAND'S FORMULA (21). FOR COAXIAL COILS OF EQUAL RADII

$$A = a = 25 \qquad b = c = 2 \text{ cm} \qquad d = 10 \qquad \text{(Fig. 12.)}$$

The mutual inductance of the two coils is $\frac{M}{n_1 n_2} = M_0 + \Delta M$.

We find M_0 by formula 1, 8, or 13, and ΔM by 21 and 22.

$$M_0 = 107.4885\pi$$

$$k = \sin\gamma = \frac{50}{\sqrt{2600}} = 0.9805807$$

$$k^2 = 0.9615383$$

$$\log_{10} F = 0.4821754$$

$$\log_{10} E = 0.0207625$$

By Table II, since $\tan\gamma = 5$, $\log F = 0.4821752$ and $\log E = 0.0207626$. These slight differences in the logarithms obtained in the two different ways amount to scarcely one part in two million of F and E, respectively, and may usually be neglected. If more accurate values are required they may be obtained by carrying the interpolations further in Legendre's table, provided the angle γ is obtained with sufficient accuracy.

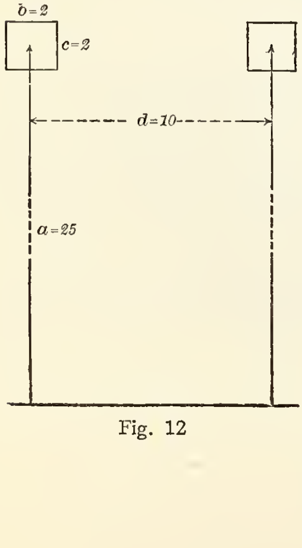

Fig. 12

Substituting these values in formula (22) we obtain

$$\frac{d^2M}{da^2} = -0.9081\pi$$

$$\frac{d^2M}{dx^2} = +1.0639\pi \qquad b^2 = c^2 = 4$$

Substituting these values in formula (21) we obtain

$$\Delta M = .05194\pi$$

$$\therefore \frac{M}{n_1 n_2} = M_0 + \Delta M = (107.4885 + 0.0519)\pi$$

$$= 337.8481 \text{ cm.}$$

The correction ΔM thus amounts to about 1 part in 2000 of M. At a distance $d=20$ cm, the correction is over 1 part in 1000. For a coil of section 4×4 cm at $d=10$, ΔM would be four times as large as the value above, or about one part in five hundred, and at 20 cm one part in two hundred and fifty.

EXAMPLE 21. ROWLAND'S FORMULA (20). FOR COILS OF EQUAL RADII BUT UNEQUAL SECTION

Let us take $a=25$, $d=10$ as in the preceding example, but instead of supposing the sections of the coils to be equal let us take

$$b_1=4 \qquad b_2=2$$
$$c_1=1 \qquad c_2=2$$

The values of $\frac{d^2M_0}{dx^2}$ and $\frac{d^2M_0}{da^2}$ will be the same as in the preceding example. Substituting these in (20) we find $\Delta M=0.69474\pi$

$$M_0=107.4885\pi$$
$$\Delta M=\underline{\quad 0.6974\pi}$$
$$\therefore \frac{M}{n_1n_2}=108.1859\pi=339.8761 \text{ cm.}$$

The correction here is fourteen times as great as in the previous example, although the areas of the cross sections of the two coils are the same as in the preceding case.

EXAMPLE 22. RAYLEIGH'S FORMULA (24). FOR COAXIAL COILS OF EQUAL RADII

$$A=a=25 \qquad b=4 \qquad c=1 \qquad d=10$$

We now find by formula (1) in accordance with formula (24) the mutual inductance of the following pairs of circles (Fig. 13):

O, 1 when $a=25$, $A=25.5$, $d=10$; O, 4 when $a=25$, $A=24.5$, $d=10$; O, 2 when $a=A=25$ and $d=8$; O, 3 when $A=a=25$, $d=12$ and finally O, O′ when $A=a=25$, $d=10$. Thus:

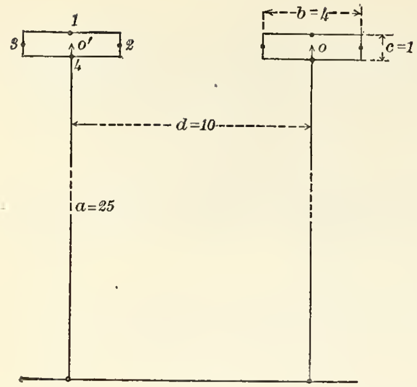

Fig. 13

$$
\begin{aligned}
M_1 &= 109.3217\pi \\
M_4 &= 105.4287\pi \\
M_2 &= 127.3949\pi \\
M_3 &= \underline{91.9206\pi} \\
& 434.0659\pi \\
M_0 &= \underline{107.4885\pi} \\
& 326.5774\pi \\
\therefore M &= 108.8591\pi \\
M_0 &= \underline{107.4885\pi} \\
\Delta M &= 1.3706\pi \text{ cm.}
\end{aligned}
$$

EXAMPLE 23. RAYLEIGH'S FORMULA (23). COILS OF UNEQUAL RADII AND UNEQUAL SECTION

Let

$$
\begin{array}{llll}
A = 25 & b_1 = 4 & c_1 = 1 & d = 10 \\
a = 20 & b_2 = 2 & c_2 = 3 &
\end{array}
$$

We have then to calculate the mutual inductances of the following pairs of circles:

	A	a	d		A	a	d
M_1	24.5	20	10	M_6	25	20	11
M_2	25	20	8	M_7	25	21.5	10
M_3	25.5	20	10	M_8	25	20	9
M_4	25	20	12	M_0	25	20	10
M_5	25	18.5	10				

These have been calculated by means of Havelock's formula (16), with the following results:

$M_1 =$ 248.41280
$M_2 =$ 28[illegible].04027
$M_3 =$ [illegible]8.77440 $\qquad M_0 = 248.7873$
$M_4 =$ 214.75755
$M_5 =$ 216.60185
$M_6 =$ 231.04386
$M_7 =$ 279.81417
$M_8 =$ 268.09410

Sum $=$ 1996.5390
2 M_0 $=$ 497.5746
Diff $=$ 1498.9644

$$\frac{1}{6}\text{ Diff.} = 249.8272 = \frac{M}{n_1 n_2}$$

EXAMPLE 24. LYLE'S FORMULA (27). FOR COILS OF SQUARE SECTION

$$A = a = 25 \text{ cm} \qquad b = c = 2 \text{ cm} \qquad d = 10 \text{ cm.}$$

The equivalent radius $r = a\left(1 + \frac{b^2}{24a^2}\right)$

$$r = 25\left(1 + \frac{4}{15000}\right) = 25.00667 \text{ cm.}$$

M is now found by using formula 1, 8, or 13, employing r in place of a as the radius.

The result is $M = 337.8475$, agreeing very closely with the result found under example 20.

$$M - M_0 = \Delta M = .0517\pi$$

EXAMPLE 25. LYLE'S FORMULA (28). FOR COILS OF RECTANGULAR SECTION

$$A = a = 25 \qquad b = 4 \qquad c = 1 \qquad d = 10$$

$$r = 25\left(1 + \frac{1}{15000}\right) = 25.00167$$

$\beta^2 = \frac{b^2 - c^2}{12} = \frac{15}{12} = 1.25$, $2\beta = 2.236$ cm, the distance apart of the two filaments which replace the coil (Fig. 14). We now find by formula (1), (8), or (13) the mutual inductances of two circles 1, 2 on the two circles 3, 4, where $a = 25.00167$ and d is 7.764, 10 and 12.236 cm, respectively. Thus:

$$\begin{aligned}
2\,M_{13} &= 215.00228\pi \\
M_{14} &= 90.31304\pi \\
M_{23} &= \underline{130.14060\pi} \\
4\,M &= 435.45592\pi \\
\therefore M &= 108.8640\ \pi \\
M_0 &= \underline{107.4885\ \pi} \\
\Delta M &= 1.3735\ \pi
\end{aligned}$$

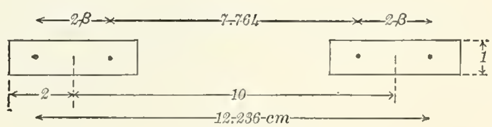

Fig. 14

ΔM = the correction for section of the coils whose dimensions are given above. These values of M and ΔM agree nearly with the results obtained in example 22 above.

EXAMPLE 26. LYLE'S FORMULA (28). FOR UNEQUAL COILS OF RECTANGULAR SECTION

Let us take the same coils as in example 23

$$A = 25 \qquad b_1 = 4 \qquad c_1 = 1 \qquad d = 10$$
$$a = 20 \qquad b_2 = 2 \qquad c_2 = 3$$

For the first coil we find

$$r = 25\left(1 + \frac{c_1^{\,2}}{24A^2}\right) = 25.001667 \text{ cm}$$
$$\beta = 1.118034 \text{ cm}$$

For the second coil

$$r = 20\left(1 + \frac{b_1^{\,2}}{24a^2}\right) = 20.008333 \text{ cm}$$
$$\delta = 0.645497$$
$$r + \delta = 20.653830$$
$$r - \delta = 19.362836$$

We then calculate the mutual inductance of the following pairs of circles:

	A	a	d
M_{13}	25.001667	20.653830	11.118034
M_{14}	"	19.362836	11.118034
M_{23}	"	20.653830	8.881966
M_{24}	"	19.362836	8.881966

The results by Havelock's formula (16) were

$$M_{13} = 241.29369$$
$$M_{14} = 216.91302$$
$$M_{23} = 286.13490$$
$$M_{24} = 255.03471$$
$$\text{Sum} = 999.37632$$
$$\frac{1}{4}\text{ Sum} = 249.8441 = \frac{M}{n_1 n_2}$$

which differs from the value by Rayleigh's formula (23) by six or seven in a hundred thousand.

A more accurate value would, in each case, be found if each coil were subdivided and the formulas applied to each of the components as described on page 43. Such a proceeding is, however, rather tedious, although necessary in precise work.

EXAMPLE 27. ROSA'S FORMULA (29). FOR COILS OF EQUAL RADII

$$A=a=25 \qquad b=4 \qquad c=1 \qquad d=10$$

(same coils as examples 22, 25).

$$\log_e\frac{8a}{d}=\log_e 20=2.9957$$

$$\begin{array}{rcrl}
\dfrac{3b^2+c^2}{96a^2}\cdot\log_e\dfrac{8a}{d}=\dfrac{49\times 2.9957}{60000} & = & .0024465 & \\
\dfrac{b^2-c^2}{12d^2}=\dfrac{15}{1200} & = & .0125000 & \\
\dfrac{2b^4+2c-5b^2c^2}{120d^4}=\dfrac{434}{1200000} & = & .0003617 & \\
\dfrac{3b^6-3c^6+14b^2c^4-14b^4c^2}{504d^6}=\dfrac{8925}{504\times 10^6} & = & .0000177 & \\
\dfrac{6b^4+6c^4+5b^2c^2}{5760a^2d^2}=\dfrac{1622}{360\times 10^6} & = & .0000045 & \\
\dfrac{7c^2d^2}{1024a^4}\left(\log_e\dfrac{8a}{d}-\dfrac{163}{84}\right) & = & \underline{.0000018} & .0153322 \\
-\dfrac{11b^2-3c^2}{192a^2}= -\dfrac{173}{120000} & = & -.0014417 & \\
-\dfrac{15b^2d^2}{1024a^4}\left(\log_e\dfrac{8a}{d}-\dfrac{97}{60}\right) & = & -\underline{.0000827} & -\underline{.0015244} \\
 & & & .0138078
\end{array}$$

$$4a=100, \quad \therefore \quad \Delta M=1.3808\ \pi \text{ cm.}$$

This is a little larger value than found by formulas (24) and (28), and we shall see later that it is more nearly correct than either of the other values.

EXAMPLE 28. ROSA'S FORMULAS (30) AND (31). FOR COILS OF EQUAL RADII AND SQUARE SECTION

$$A=a=25 \qquad b=c=2 \qquad d=10$$

$$\begin{array}{rcrcrr}
\log_e\dfrac{8a}{d}-1 & = & 2.9957-1 & = & 1.9957 & \\
\dfrac{17b^2}{240d^2} & = & \dfrac{68}{24000} & = & \underline{.0028} & 1.9985 \\
\dfrac{-a^2b^2}{5d^4} & = & -\dfrac{2500}{50000} & = & -\ .0500 & \\
\dfrac{-3d^2}{16a^2}\left(\log_e\dfrac{8a}{d}-\dfrac{4}{3}\right) & = & -\dfrac{300\times 1.6624}{10000} & = & -\ \underline{.0499} & -\underline{.0999} \\
 & & & & & 1.8986
\end{array}$$

$$\frac{b^2}{6a}=\frac{4}{150} \quad \therefore \quad \Delta M=.05063\pi$$

The approximate formula (31) would have given .0519 (agreeing with formulas 21 and 27), which would be amply accurate for any experimental purpose. When the section is larger these small terms are, however, more important.

EXAMPLE 29. SECOND EXAMPLE BY FORMULA (30)

$$A=a=25 \qquad b=c=5 \qquad d=10$$

$$\begin{array}{rcll}
\log_e \dfrac{8a}{d}-1 & = & 1.9957 & \\
\dfrac{17b^2}{240d^2} & = & \underline{.0177} & 2.0134 \\
\dfrac{-a^2b^2}{5d^4} & = & -.3125 & \\
\dfrac{-3d^2}{16a^2}\left(\log_e \dfrac{8a}{d}-\dfrac{4}{3}\right) & = & -\underline{.0499} & -\underline{.3624} \\
 & & & 1.6510
\end{array}$$

$$\frac{b^2}{6a}=\frac{25}{150}$$

$$\therefore \Delta M=0.2752\pi$$

$$\underline{M_0=107.4885\pi} \text{ (see example 20)}$$

$$\therefore \frac{M}{n_1n_2}=107.7637\pi \text{ cm.}$$

This is a very simple formula for computing ΔM, and within a considerable range (i. e., d not larger than a and yet the coils not in contact) it is very accurate.

EXAMPLE 30. FORMULA (32). COILS OF EQUAL RADII, BUT UNEQUAL SECTION

For this we will take the coils of example 21

$$\begin{array}{ll}
a=25 & d=10 \\
b_1=4 & b_2=2 \\
c_1=1 & c_2=2
\end{array}$$

$$
\begin{aligned}
\text{1st term} &= 0.0016227\\
\text{2d} \quad \text{“} &= -0.0008542\\
\text{3d} \quad \text{“} &= 0.0062500\\
\text{4th} \quad \text{“} &= \underline{0.0000570}\\
\text{Sum} &= 0.0070755\\
\therefore \Delta M &= 0.70755\pi\\
M_0 &= \underline{107.4885\pi}\\
\text{Sum} &= 108.1960\pi\\
\therefore \frac{M}{n_1 n_2} &= 339.9078 \text{ cm.}
\end{aligned}
$$

This example shows that the fourth differentials neglected in (20) here amount to one part in ten thousand.

EXAMPLE 31. ROSA-WEINSTEIN FORMULA (33). FOR COILS OF EQUAL RADII AND EQUAL SECTION

$$a=25 \qquad b=4 \qquad c=1 \qquad d=10$$

$$\alpha_1 = 15.0000533 \qquad \sin^2\gamma = \frac{2500}{2600} = \frac{25}{26}$$

$$\alpha_2 = 0.0020267 \qquad \cos^2\gamma = \frac{100}{2600} = \frac{1}{26}$$

$$\alpha_3 = 0.2170000 \qquad \frac{c^2}{24a^2} = .0000667$$

$$\alpha_1 - \alpha_2 - \alpha_3 + (2\alpha_2 - 3\alpha_3)\cos^2\gamma + 8\alpha_3\cos^4\gamma = 14.7587120$$

$$\alpha_1 + \frac{\alpha_2}{2} + 2\alpha_3 + (2\alpha_2 + 3\alpha_3)\cos^2\gamma + 8\alpha_3\cos^4\gamma = 15.4628292$$

$$A = 0.0004730 \qquad \text{Also } F = 3.0351168$$

$$B = 0.0123901 \qquad E = 1.0489686$$

$$
\begin{aligned}
(F-E)\left(A + \frac{c^2}{24a^2}\right) &= 0.0010719\\
EB &= 0.0129968\\
\text{Sum} &= 0.0140687
\end{aligned}
$$

$$4\pi a \sin\gamma = 100\pi\sqrt{\frac{25}{16}} \quad \therefore \Delta M = 1.3795\pi \text{ cm.}$$

This is not as simple to calculate as (29) and when d is less than $a/2$ is less accurate than (29). But for $d=a$ or greater it is more accurate than (29), and indeed the most accurate of all the formulas.

EXAMPLE 32. FORMULA (34). MUTUAL INDUCTANCE IN TERMS OF SELF-INDUCTANCE. FOR COILS RELATIVELY NEAR

For $a=25$, $b=1$, $c=1$, we have, n being the number of turns in one of the two equal coils,

$$L_1 = 4\pi a n^2\ (4.103816)$$

For $b=2$, $c=1$,

$$L_2 = 4\pi a n^2\ (4 \times 3.698695)$$

For $b=3$, $c=1$,

$$L = 4\pi a n^2\ (9 \times 3.411766)$$

Then the mutual inductance of 1 on 3 is by formula (34)

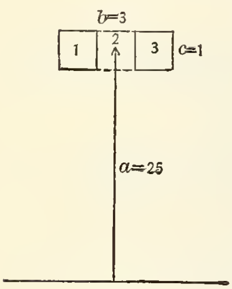

Fig. 15

$$\begin{aligned} M &= 4\pi a n^2 \left[\frac{L + L_1 - 2L_2}{2} \right] \\ &= 4\pi a n^2 \left[\frac{30.705894 + 4.103816 - 29.589560}{2} \right] \\ &= 4\pi a n^2 \times 2.610075 \\ &= 819.979\ n^2 \text{ cm.} \end{aligned}$$

If $n = 100$,

$$M = 8.19979 \text{ millihenrys,}$$

as the mutual inductance of coil 1 on coil 3, Fig. 15.

EXAMPLE 33. FORMULA (35). MUTUAL INDUCTANCE BY GEOMETRICAL MEAN DISTANCE

$$\begin{aligned} A &= 25.1 \\ a &= 25.0 \\ b = c &= 0.1 \text{ cm} \\ d &= 0.1 \text{ cm.} \end{aligned}$$

The geometrical mean distance of two coils, corner to corner, as in Fig. 10, is 0.997701, and $\log \frac{r}{R} = 0.002302$

$$\begin{aligned} \therefore\ \Delta M &= 100 \times 0.002302\ (1.002)\pi \\ &= 0.2307\pi \text{ cm.} \end{aligned}$$

3. MUTUAL INDUCTANCE OF COAXIAL SOLENOIDS

There are several formulas for the calculation of the mutual inductance of coaxial solenoids. Although few of these formulas

are exact, the approximate formulas often permit inductances to be calculated with very great accuracy by using a sufficient number of terms of the series by which they are expressed.

CONCENTRIC, COAXIAL SOLENOIDS OF EQUAL LENGTH

MAXWELL'S FORMULA [29]

The mutual inductance M of two coaxial solenoids of equal length (Fig. 16) is given by the following expression, due to Maxwell, where A and a are the radii of the outer and inner solenoids, respectively, l is the common length, and n_1 and n_2 the number of turns of wire per cm on the single layer winding of the outer and inner solenoids, respectively:

$$M = 4\pi^2 a^2 n_1 n_2 [l - 2A\alpha]$$

where

$$r = \sqrt{l^2 + A^2}$$

$$\begin{aligned}
\alpha = {} & \frac{A-r+l}{2A} - \frac{a^2}{16A^2}\left(1 - \frac{A^3}{r^3}\right) - \frac{a^4}{64A^4}\left(\frac{1}{2} + 2\frac{A^5}{r^5} - \frac{5}{2}\frac{A^7}{r^7}\right) \\
& - \frac{35}{2048}\frac{a^6}{A^6}\left(\frac{1}{7} - \frac{8}{7}\frac{A^7}{r^7} + 4\frac{A^9}{r^9} - 3\frac{A^{11}}{r^{11}}\right) \\
& - \frac{1}{2}\frac{63}{128^2}\frac{a^8}{A^8}\left(\frac{5}{9} + \frac{64}{9}\frac{A^9}{r^9} - 48\frac{A^{11}}{r^{11}} + 88\frac{A^{13}}{r^{13}} - \frac{143}{3}\frac{A^{15}}{r^{15}}\right) \\
& - \frac{231}{512^2}\frac{a^{10}}{A^{10}}\left(\frac{7}{11} - \frac{128}{11}\frac{A^{11}}{r^{11}} + 128\frac{A^{13}}{r^{13}} - 416\frac{A^{15}}{r^{15}} + 520\frac{A^{17}}{r^{17}} - 221\frac{A^{19}}{r^{19}}\right) \\
& - \frac{1}{2}\frac{429}{1024^2}\frac{a^{12}}{A^{12}}\left(\frac{21}{13} + \frac{512}{13}\frac{A^{13}}{r^{13}} - 640\frac{A^{15}}{r^{15}} + 3200\frac{A^{17}}{r^{17}} - 6800\frac{A^{19}}{r^{19}}\right. \\
& \qquad \left. + 6460\frac{A^{21}}{r^{21}} - 2261\frac{A^{23}}{r^{23}}\right) \qquad [36] \\
& - \frac{6435}{8192^2}\frac{a^{14}}{A^{14}}\left(\frac{11}{5} - \frac{1024}{15}\frac{A^{15}}{r^{15}} + 1536\frac{A^{17}}{r^{17}} - 10880\frac{A^{19}}{r^{19}} + \frac{103360}{3}\frac{A^{21}}{r^{21}}\right. \\
& \qquad \left. - 54264\frac{A^{23}}{r^{23}} + \frac{208012}{5}\frac{A^{25}}{r^{25}} - \frac{37145}{3}\frac{A^{27}}{r^{27}}\right) - \dots\dots\dots
\end{aligned}$$

[29] Electricity and Magnetism, Vol. II, § 678.

Putting

$$M = M_0 - \Delta M$$

$M_0 = 4\pi^2 a^2 n_1 n_2 l$ is the mutual inductance of an infinite outer solenoid and the finite inner solenoid, while ΔM is the correction due to the ends.

Equation (36) is Maxwell's expression, except that we have carried it out much further than Maxwell did. We would, however, emphasize that in the great majority of cases only three or four terms need be calculated in α, and in these only the first few terms in each parenthesis, to obtain a satisfactory accuracy.

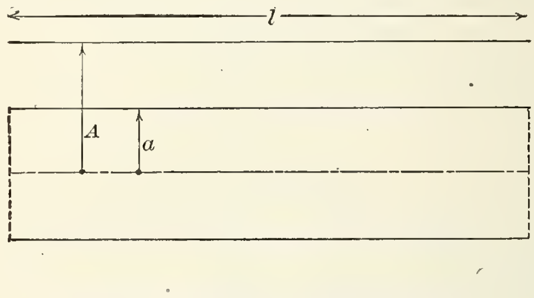

Fig. 16

Since, however, this formula is the most valuable single expression known for the case of solenoids of equal length, it has seemed advisable to extend the series far enough to take care of the most unfavorable cases, which may arise in practice. At the same time the extra terms found have proved of use in checking our extension of Ròiti's formula below.

It should be noticed that the algebraic sums of the coefficients in each of the parentheses is equal to zero. For very long coils $\left(\frac{A}{r}\text{ small}\right)$ the quantities in the parentheses are sensibly equal to the absolute term inside. For very short coils the parentheses are a little larger, reaching a maximum in the region $\frac{A}{r} = 0.9$, but falling abruptly to zero at the limit $\frac{A}{r} = 1$. The expression for α is

therefore rapidly convergent for coils of all lengths, even when the inner radius is nearly as great as the outer radius. In such cases the number of terms to be calculated in the above formula may become considerable, but even then it is simpler to use this series than to make the calculation with an absolute formula, such as those of Cohen or Nagaoka.

Equation (36) shows that the mutual inductance is proportional to $l-2A\alpha$; or the length l must be reduced by $A\alpha$ on each end. When a is small and l is large, α is $1/2$ approximately. That is, the length l is reduced by A, the radius of the outer solenoid.

For the case of two coils each of more than one layer the above formula may be used, A and a being the mean radii, and n_1 and n_2 the total number of turns per cm in all the layers. The result will be only approximate, but usually less in error than if one uses the formula of Maxwell § 679 quoted by Mascart and Joubert.[30]

When the solenoids are very long in comparison with the radii, formula (36) may be simplified by omitting the terms in A/l, A^3/r^3, A^5/r^5, etc. The expression for α then becomes

$$\alpha=\frac{1}{2}-\frac{a^2}{16A^2}-\frac{a^4}{128A^4}-\frac{5a^6}{2048A^6}-\cdots \qquad [37]$$

Heaviside[31] gives an extension of formula (37), but as it neglects $\frac{A}{l}, \frac{A^3}{r^3}$, etc., the additional terms are of no importance, being smaller than the terms already neglected in (37).

HAVELOCK'S FORMULA[32]

This formula for coaxial, concentric solenoids of equal length bears a close resemblance to the preceding, the main difference being that here l enters in place of the quantity $r=\sqrt{l^2+A^2}$ in

[30] Electricity and Magnetism, Vol. I, p. 533.

[31] There are some misprints in Heaviside, 2, p. 277. The radius of the *inner* solenoid should be c_2, of the *outer* c_1, and ρ is c_2^2/c_1^2.

[32] Phil. Mag., 15, p. 339; 1908. There is a misprint in Havelock's equation (25). In the factor outside the brackets, read a^2 instead of a.

equation (36). Using the same notation as in the latter this formula reads:

$$M = 4\pi^2 a^2 n_1 n_2 [l - 2A\beta]$$

where

$$\beta = \left[\frac{1}{2} - \frac{1}{16}\frac{a^2}{A^2} - \frac{1}{128}\frac{a^4}{A^4} - \frac{5}{2048}\frac{a^6}{A^6} - \frac{35}{32768}\frac{a^8}{A^8} - \cdots \right.$$
$$- \frac{1}{4}\frac{A}{l} + \frac{1}{16}\left(1 + \frac{a^2}{A^2}\right)\left(\frac{A}{l}\right)^3 - \frac{1}{32}\left(1 + \frac{3a^2}{A^2} + \frac{a^4}{A^4}\right)\left(\frac{A}{l}\right)^5$$
$$\left. + \frac{5}{256}\left(1 + 6\frac{a^2}{A^2} + 6\frac{a^4}{A^4} + \frac{a^6}{A^6}\right)\left(\frac{A}{l}\right)^7 - \cdots \right] \qquad [38]$$

Havelock gives the expressions for the general terms in $\frac{A}{a}$ and $\frac{A}{l}$, so that the computation of β may be carried out so as to include terms of higher order when necessary. These expressions are

$$-\frac{(2n-1)[1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-3)]^2}{2^{2n+1} n!(n+1)!}\left(\frac{a}{A}\right)^{2n}$$

and

$$\frac{(-1)^s (2s)!\, F\left(-s-1, -s, 2, \frac{a^2}{A^2}\right)}{2^{2s+2} s!\,(s+1)!}\left(\frac{A}{l}\right)^{2s+1}$$

where F is a hypergeometric series in $\frac{a^2}{A^2}$, all of whose terms after that in $\left(\frac{a}{A}\right)^{2s}$ are zero.

$$F(\alpha,\beta,\gamma,z) = 1 + \frac{\alpha\beta}{1.\gamma}z + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1 \cdot 2 \cdot \gamma(\gamma+1)}z^2$$
$$+ \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1 \cdot 2 \cdot 3 \cdot \gamma(\gamma+1)(\gamma+2)}z^3 + \cdots$$

Formula (38) may be regarded as intermediate between (36) and (37), being applicable only to coils whose length is greater than the radius of the larger coil. In such cases, however, it furnishes a valuable check on Maxwell's formula.

CONCENTRIC COAXIAL SOLENOIDS, INNER COIL SHORTER THAN THE OUTER

RÒITI'S FORMULA

For a pair of concentric, coaxial solenoids of which the inner solenoid is shorter than the outer, we have the following:[33]

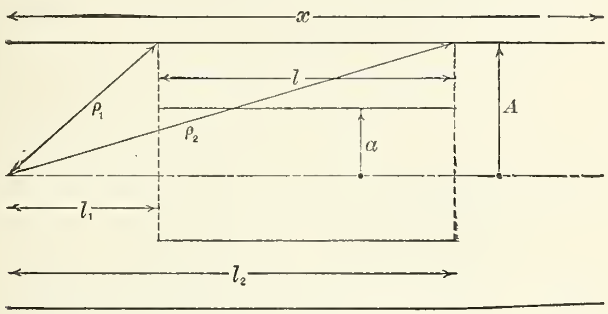

Fig. 17

$$M=4\pi^2a^2n_1n_2\Big[\rho_2-\rho_1+\frac{a^2A^2}{8}\left(\frac{1}{\rho_1^3}-\frac{1}{\rho_2^3}\right)-\frac{a^4A^2}{16}\left(\frac{1}{\rho_1^5}-\frac{1}{\rho_2^5}\right)$$
$$+\frac{5}{64}a^4A^4\left(1+\frac{1}{2}\frac{a^2}{A^2}\right)\left(\frac{1}{\rho_1^7}-\frac{1}{\rho_2^7}\right)$$
$$-\frac{35}{256}a^6A^4\left(1+\frac{1}{5}\frac{a^2}{A^2}\right)\left(\frac{1}{\rho_1^9}-\frac{1}{\rho_2^9}\right)$$
$$+\frac{105}{1024}a^6A^6\left(1+\frac{9}{5}\frac{a^2}{A^2}+\frac{1}{5}\frac{a^4}{A^4}\right)\left(\frac{1}{\rho_1^{11}}-\frac{1}{\rho_2^{11}}\right) \qquad [39]$$
$$-\frac{693}{2048}a^8A^6\left(1+\frac{2}{3}\frac{a^2}{A^2}+\frac{1}{21}\frac{a^4}{A^4}\right)\left(\frac{1}{\rho_1^{13}}-\frac{1}{\rho_2^{13}}\right)$$
$$+\frac{3003}{16384}a^8A^8\left(1+4\frac{a^2}{A^2}+\frac{10}{7}\frac{a^4}{A^4}+\frac{1}{14}\frac{a^6}{A^6}\right)\left(\frac{1}{\rho_1^{15}}-\frac{1}{\rho_2^{15}}\right)-\cdots$$

in which (see Fig. 17)

---

[33] For the derivation and method of extension of this formula see this Bulletin, 3, pp. 309-310. Recently we have carried it out still further to include the case of coils of moderate length. This formula was originally given (without proof and including the main term in $\left(\frac{1}{\rho_1^7}-\frac{1}{\rho_2^7}\right)$ only) in this Bulletin, 2, p. 130; 1906.

$$\rho_1=\sqrt{l_1^2+A^2} \text{ where } l_1=\frac{x-l}{2}$$

$$\rho_2=\sqrt{l_2^2+A^2} \quad \text{"} \quad l_2=\frac{x+l}{2}$$

$l=l_2-l_1=$ length of inner solenoid.

$x=$ length of outer solenoid and A and a the radii.

When $\frac{l}{x}$ is small (case of short inner coil), $(\rho_2-\rho_1)$ is most accurately calculated by the exact formula $(\rho_2-\rho_1)=\frac{xl}{\rho_1+\rho_2}$, the denominator being calculated from the above expressions for ρ_1 and ρ_2.

For long coils $\left(\frac{2A}{x}\text{ small}\right)$ the above formula is rapidly convergent, especially if the inner coil is considerably shorter than the outer. This formula may also be used for short coils $\left(\frac{x}{2A}\text{ small}\right)$, the convergence being most rapid when the radius of the inner coil is small in comparison with that of the outer. For very short coils, we have expanded formula (39) in a series in ascending powers of $\frac{a^2}{A^2}$. This formula is, however, not so accurate, nor so simple to use as that of Searle and Airey, and has not been included in this collection.

A peculiarity of Ròiti's formula is that the successive terms, especially in the case of short coils, are nearly equal in pairs. Thus the terms in $\left(\frac{1}{\rho_1^5}-\frac{1}{\rho_2^5}\right)$ and $\left(\frac{1}{\rho_1^7}-\frac{1}{\rho_2^7}\right)$ are of the same order of magnitude, but of opposite sign; similarly for the terms involving the ninth and eleventh powers of ρ_1 and ρ_2, and so on. For the limiting case $x=l$, Ròiti's formula goes over into Maxwell's (36), as would be expected, since both are derived by integration of the same original expression between appropriate limits. To obtain, however, the same precision, twice as many terms have to be calculated in Ròiti's formula as in Maxwell's. We see from these considerations, that in using Ròiti's formula, the inner coil need not be very different in length from the outer coil, although in general the convergence is better with a relatively short inner solenoid.

GRAY'S FORMULA

Gray[34] gives a general expression for the mutual kinetic energy of two solenoidal coils which may or may not be concentric, and their axes may be at any angle ϕ. The most important case in practice is when the two coils are coaxial. In that case the zonal harmonic factors in each term reduce to unity, and half the terms become zero. Putting the current in each equal to unity, the mutual kinetic energy becomes the mutual inductance M.

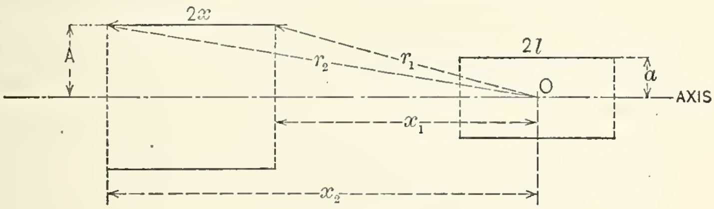

Fig. 18

Let $2x$, A, n_1 be respectively the length, radius, and number of turns per cm of one of the coils, and $2l$, a, n_2 be the corresponding quantities for the other solenoid. Let, further, x_1 and x_2 be the distances, along the axis, between the center of the coil with radius a and the nearer and further end planes, respectively, of the coil with radius A, and let r_1 and r_2 be the diagonals (Fig. 18).

$$r_1 = \sqrt{x_1^2 + A^2} \qquad r_2 = \sqrt{x_2^2 + A^2}$$

Gray's expression with these changes becomes

$$M = \pi^2 a^2 A^2 n_1 n_2 [K_1 k_1 + K_3 k_3 + K_5 k_5 + \cdot \cdot \cdot] \qquad [40]$$

where K_1, K_3, etc., are functions of x and A, and k_1, k_3, etc., are functions of l and a.[35]

[34] Absolute Measurements, 2, Part I, p. 274, equation 53.
[35] Rosa, this Bulletin, 3, p. 221; 1907.

$$K_1=\frac{2}{A^2}\left(\frac{x_2}{r_2}-\frac{x_1}{r_1}\right)$$

$$K_3=\frac{1}{2}\left(\frac{x_1}{r_1^5}-\frac{x_2}{r_2^5}\right)$$

$$K_5=-\frac{A^2}{8}\left\{\frac{x_1}{r_1^9}\left(3-4\frac{x_1^2}{A^2}\right)-\frac{x_2}{r_2^9}\left(3-4\frac{x_2^2}{A^2}\right)\right\}$$

$$=-\frac{A^2}{8}\left\{\frac{x_1}{r_1^9}X_2'-\frac{x_2}{r_2^9}X_2''\right\} \qquad [40a]$$

$$K_7=\frac{A^4}{8}\left\{\frac{x_1}{r_1^{13}}\left(\frac{5}{2}-10\frac{x_1^2}{A^2}+4\frac{x_1^4}{A^4}\right)-\frac{x_2}{r_2^{13}}\left(\frac{5}{2}-10\frac{x_2^2}{A^2}+4\frac{x_2^4}{A^4}\right)\right\}$$

$$=\frac{A^4}{8}\left\{\frac{x_1}{r_1^{13}}X_4'-\frac{x_2}{r_2^{13}}X_4''\right\}$$

. . . .

$$k_1=2l$$

$$k_3=-a^2l\left(3-4\frac{l^2}{a^2}\right)=-a^2lL_2$$

$$k_5=a^4l\left(\frac{5}{2}-10\frac{l^2}{a^2}+4\frac{l^4}{a^4}\right)=a^4lL_4$$

$$k_7=-a^6l\left(\frac{35}{16}-\frac{35}{2}\frac{l^2}{a^2}+21\frac{l^4}{a^4}-4\frac{l^6}{a^6}\right)=-a^6lL_6$$

. . . .

This formula is simple and convenient for calculation, if only a few terms need be evaluated. This is the case when r_1 and r_2 are large (coils relatively far apart). The coefficients L_{2n}, X'_{2n} and X''_{2n} are derived from the same polynomial S_{2n} by substituting $\frac{l^2}{a^2}$, $\frac{x_1^2}{A^2}$, and $\frac{x_2^2}{A^2}$, respectively.

For short coils relatively far apart these polynomials are all small. Table XVIII gives values of the polynomial S_{2n} with varying argument, to aid in calculations where great accuracy is not desired, or to aid in making preliminary calculations to see whether the convergence will be satisfactory in any particular case.

If the coils are concentric, and the ratio of the length of the winding of the outer coil to the radius is $\sqrt{3}$ to 1, $K_5=0$, and if the same condition holds for the inner coil, $k_3=0$. If in addition a is considerably smaller than A, the terms of higher order become negligible and (40) reduces to

$$M=\frac{2\pi^2a^2N_1N_2}{d} \qquad [41]$$

where d is half the diagonal of the outer coil, $=\sqrt{x^2+A^2}$. When the dimensions depart slightly from these theoretical ratios the small correction terms to (41) can be calculated.[35a] The general case for concentric coils is treated in the next section.

SEARLE AND AIREY'S FORMULA

The following expression for the mutual inductance of two concentric, coaxial solenoidal coils (Fig. 19) has been given by Searle and Airey:[36]

$$M=g_1G_1+g_3G_3+g_5G_5+g_7G_7+\cdots$$
$$=\frac{2\pi^2a^2N_1N_2}{d}\left[1-\frac{A^2}{2d^4}\frac{4l^2-3a^2}{4}-\frac{A^2(4x^2-3A^2)}{8d^8}\cdot\frac{8l^4-20l^2a^2+5a^4}{8}\right.$$
$$\left.-\frac{A^2(8x^4-20x^2A^2+5A^4)}{16d^{12}}\cdot\frac{(64l^6-336l^4a^2+280l^2a^4-35a^6)}{64}-\cdots\right] \quad [42]$$

The notation of (42) differs slightly from that used by Searle and Airey.

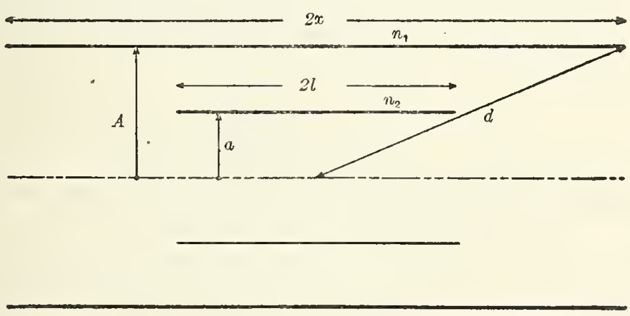

Fig. 19

Equation (42) has been extended and put for greater convenience in calculation into the form[37] shown on next page.

[35a] Rosa, this Bulletin, **3**, p. 221; 1907.

[36] The Electrician (London), **56**, p. 318; 1905.

[37] Rosa, this Bulletin, **3**, p. 224; 1907.

$$M=\frac{2\pi^2a^2N_1N_2}{d}\left[1+\frac{A^2a^2}{8d^4}L_2+\frac{A^4a^4}{32d^8}X_2L_4\right.$$
$$+\frac{A^6a^6}{32d^{12}}X_4L_6+\frac{A^8a^8}{32d^{16}}X_6L_8+\cdots$$
$$\left.+\frac{1}{32}\left(\frac{Aa}{d^2}\right)^{2n}X_{2n-2}L_{2n}+\cdots\right] \qquad [43]$$

where

$$X_2=3-4\frac{x^2}{A^2} \qquad L_2=3-4\frac{l^2}{a^2} \qquad d=\sqrt{x^2+A^2}$$

$$X_4=\frac{5}{2}-10\frac{x^2}{A^2}+4\frac{x^4}{A^4} \qquad L_4=\frac{5}{2}-10\frac{l^2}{a^2}+4\frac{l^4}{a^4}$$

$$X_6=\frac{35}{16}-\frac{35}{2}\frac{x^2}{A^2}+21\frac{x^4}{A^4}-4\frac{x^6}{A^6} \qquad L_6=\frac{35}{16}-\frac{35}{2}\frac{l^2}{a^2}+21\frac{l^4}{a^4}-4\frac{l^6}{a^6}$$

$$X_8=\frac{63}{32}-\frac{105}{4}\frac{x^2}{A^2}+63\frac{x^4}{A^4}-36\frac{x^6}{A^6}+4\frac{x^8}{A^8} \qquad L_8=\frac{63}{32}-\frac{105}{4}\frac{l^2}{a^2}+63\frac{l^4}{a^4}-36\frac{l^6}{a^6}+4\frac{l^8}{a^8}$$

$$L_{10}=\frac{231}{128}-\frac{1155}{32}\frac{l^2}{a^2}+\frac{1155}{8}\frac{l^4}{a^4}-165\frac{l^6}{a^6}+55\frac{l^8}{a^8}-4\frac{l^{10}}{a^{10}}$$

$N_1=2xn_1$ and $N_2=2ln_2$ are the total number of turns on the two solenoids. This formula reduces to (41) when the terms after the first are negligible, as they are when the conditions assumed for (41) are fulfilled. The above expressions for L_2, X_2 show what these conditions are in order to make the second and third terms zero. If l^2/a^2 is slightly more or less than 3/4, (43) gives the value of the second term which is neglected in (41), etc.

The degree of convergence of Searle and Airey's formula depends primarily on the magnitude of the quantity $\frac{A^2a^2}{d^4}$; in certain cases, however, the values of the coefficients become of equal importance, making it necessary to examine carefully into the degree of convergence of the formula, since the terms of higher order are sometimes larger than those immediately preceding. Since the X and L

coefficients are polynomials in $\frac{l^2}{a^2}$ and $\frac{x^2}{A^2}$, each one will have a finite number of roots depending on the degree of the polynomial. The values of these coefficients will therefore, with increasing $\frac{l}{a}$ or $\frac{x}{A}$, oscillate between positive and negative values, each maximum or minimum being greater than that preceding, until, for values of the argument greater than the largest root, the values of the functions increase indefinitely without limit.

For short coils $\left(\frac{x}{A}\text{ and }\frac{l}{a}\text{ small}\right)$ the coefficients will evidently be confined to moderate values, and if, further, the inner radius is small relatively to the outer, the convergence will be very rapid. For longer coils the coefficients may attain very large values, and the convergence become very unsatisfactory, in spite of the fact that $\frac{A^2a^2}{d^4}$ is, for given radii, smaller with long coils than with short coils. The conditions are so complicated that we have calculated (Table XIX) certain values of the coefficients to aid in deciding whether, in any given case, the convergence will be satisfactory or not. The values given for $\frac{x}{A}$ and $\frac{l}{a}$ less than unity will also be found useful in calculations of the mutual inductance of short coils by Searle and Airey's formula, when the highest precision is not required. Coefficients of higher order than those given above are calculated by the formula

$$L_{2n}=\sum_{p=0}^{p=n}\frac{(-1)^{n-p}(2n+1)2n(2n-1)\cdot\cdot\cdot\cdot[2n-(2p-2)]}{\left(\frac{p+1}{4}\right)2^2\cdot4^2\cdot6^2\cdot\cdot\cdot\cdot\cdot(2p)^2}\left(\frac{l}{a}\right)^{2n-2p}$$

X_{2n} is calculated by the same expression in $\frac{x}{A}$ instead of $\frac{l}{a}$.

Table XVIII includes all the positive and negative maxima as well as the zero points of the coefficients up to and including L_{14} or X_{14}, together with the values at a number of intermediate points. Although, from the nature of the case, a table to serve as the basis of accurate calculations would be somewhat bulky, those given should suffice to simplify the use of this valuable formula.

COHEN'S FORMULA[38] FOR ANY TWO COAXIAL, CONCENTRIC SOLENOIDS

This is an absolute formula for two coaxial, concentric solenoids of lengths $2l_1$ and $2l_2$, Fig. 20.

$$M=4\pi n_1 n_2(V-V_1)$$

$$\begin{aligned} V= & -(A^2-a^2)c[F\{F(k',\theta)-E(k',\theta)\}-EF(k',\theta)] \\ & +\frac{c^4-(A^2-6Aa+a^2)c^2-2(A^2-a^2)^2}{3\sqrt{(A+a)^2+c^2}}.F \\ & +\frac{2(A^2+a^2)-c^2}{3}\sqrt{(A+a)^2+c^2}.\,E-c(A^2-a^2)\frac{\pi}{2} \end{aligned} \quad [44]$$

V_1 is obtained from V by replacing c by c_1,

$$c=l_1+l_2 \qquad c_1=l_1-l_2,$$

F and E are the complete elliptic integrals of the first and second kind to modulus k, where $k^2=\dfrac{4Aa}{(A+a)^2+c^2}$

$F(k',\theta)$ and $E(k',\theta)$ are the incomplete elliptic integrals of modulus k' and amplitude θ,

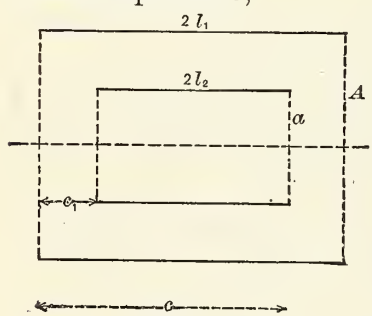

Fig. 20

$$k'^2=1-k^2=1-\frac{4Aa}{(A+a)^2+c^2}$$

$$=\frac{(A-a)^2+c^2}{(A+a)^2+c^2}$$

$$\sin^2\theta=\frac{(A^2-a^2)^2+c^2(A-a)^2}{(A^2-a^2)^2+c^2(A+a)^2}$$

NAGAOKA'S FORMULA FOR ANY COAXIAL SOLENOIDS

Nagaoka has recently given[39] an absolute formula for the mutual inductance of two coaxial solenoids, whether concentric or not, and

[38] This Bulletin, 3, p. 301; 1907.

[39] Jour. Coll. Sci., Tokyo, 27, art. 6; 1909. There are a number of misprints in Nagaoka's article, which we have detected and corrected by a careful check on the derivation of the formulas.

has expanded this in q functions in a form suitable for calculation. In the notation of Fig. 18,

$$M=4\pi n_1 n_2 Aa(I_1-I_2-I_3+I_4) \qquad [45]$$

where I_1, I_2, I_3, and I_4 are the values of the integral I, given below with the arguments

$$c=d+(x+l),\ d+(x-l),\ d-(x-l) \text{ and } d-(x+l)$$

respectively, where d is the distance between centers.

The expression for I is, in the Weierstrassian notation,

$$I=2\left\{\left(\frac{g_2}{6}-\mathfrak{p}^2 v\right)\omega_1+\mathfrak{p}v.\eta_1+\frac{\mathfrak{p}'v}{2}\left(\eta_1 v-\omega_1\frac{\sigma'}{\sigma}(v)\right)\right\}$$

where v is an auxiliary quantity, and ω_1 and g_2 are respectively the real semiperiod and invariant of the Weierstrassian function $\mathfrak{p}u$.

To calculate I, Nagaoka divides it in two parts

$$I' =\left(\frac{g_2}{6}-\mathfrak{p}^2 v\right)\omega_1+\mathfrak{p}v.\eta_1$$

$$I''=\frac{\mathfrak{p}'v}{2}\left(\eta_1 v-\omega_1\frac{\sigma'}{\sigma}(v)\right)$$

We then calculate the following auxiliary quantities

$$\alpha=\left(\frac{2}{Aa}\right)^{\frac{1}{3}}$$

$$P_1=(\mathfrak{p}v-e_1)=-\frac{c^2}{2Aa\alpha}$$

$$P_2=(\mathfrak{p}v-e_2)=\frac{(A-a)^2}{2Aa\alpha}$$

$$P_3=(\mathfrak{p}v-e_3)=\frac{(A+a)^2}{2Aa\alpha}$$

and thence (e_1-e_2), (e_1-e_3), and (e_2-e_3), which with the relation $(e_1+e_2+e_3)=0$ enable us to find $\mathfrak{p}v$.

The very small quantity q is found, as in formula (8), by the relations (see also Table XV)

$$q=\frac{l}{2}+2\left(\frac{l}{2}\right)^5+15\left(\frac{l}{2}\right)^9+\cdots$$

$$k^2=\frac{e_2-e_3}{e_1-e_3} \qquad k'^2=\frac{e_1-e_2}{e_1-e_3} \qquad l=\frac{1-\sqrt{k'}}{1+\sqrt{k'}}=\frac{k^2}{(1+k')(1+\sqrt{k'})^2}$$

and ω_1 may be calculated by any one of the following equations (the other two serving as checks):

$$\omega_1 = \frac{2\pi\sqrt{q}}{\sqrt{e_2 - e_3}}(1 + q^2 + q^6 + q^{12} + \cdots)^2$$

$$\omega_1 = \frac{\pi}{2\sqrt{e_1 - e_3}}(1 + 2q + 2q^4 + 2q^9 + \cdots)^2$$

$$\omega_1 = \frac{\pi}{2\sqrt{e_1 - e_2}}(1 - 2q + 2q^4 - 2q^9 + \cdots)^2$$

The term I' is now given by either of the following two formulas, which give a check on the calculation:

$$I' = -\left\{\frac{P_1(P_2 + P_3)}{2} + \frac{(P_1 + 2P_2)(e_2 - e_3)}{6}\right\}\omega_1 - \frac{pv}{4\omega_1}\frac{\theta''_3(0)}{\theta_3(0)}$$

$$I' = -\left\{\frac{P_1(P_2 + P_3)}{2} - \frac{(P_1 + 2P_3)(e_2 - e_3)}{6}\right\}\omega_1 - \frac{pv}{4\omega_1}\frac{\theta_0''(0)}{\theta_0(0)}$$

The quotients of the θ functions are easily calculated from the known value of q and the relations

$$\frac{\theta_3''(0)}{\theta_3(0)} = -\frac{8\pi^2(q + 4q^4 + 9q^9 + \cdots)}{1 + 2q + 2q^4 + 2q^9 + \cdots}$$

$$\frac{\theta_0''(0)}{\theta_0(0)} = \frac{8\pi^2(q - 4q^4 + 9q^9 - \cdots)}{1 - 2q + 2q^4 - 2q^9 + \cdots}$$

To calculate I'' we have

$$I'' = -\frac{\pi c}{8Aa}(A^2 - a^2)\left[\sqrt{\frac{b-1}{b+1}} + 4q^2\sqrt{b^2 - 1}\left\{1 - q^2(2b - 1)\right\}\right]$$

the expression in the brackets being nearly equal to unity. The quantity b is calculated from the equations

$$-b = \cos 2\pi w = \frac{s}{q}(1 + 2q^4 \cos 4\pi w + \cdots)$$

$$s = \frac{1}{2}\,\frac{\sqrt[4]{e_1 - e_3}\sqrt{P_2} - \sqrt[4]{e_1 - e_2}\sqrt{P_3}}{\sqrt[4]{e_1 - e_3}\sqrt{P_2} + \sqrt[4]{e_1 - e_2}\sqrt{P_3}}$$

first putting $\cos 2\pi w$ equal to its approximate value $\frac{s}{q}$, and then computing the small correction in $\cos 4\pi w$ from $\cos 2\pi w$, remembering that w is a pure imaginary. The correction to b thus found is often negligible.

The term I'' becomes less important as the difference of the radii of the solenoids becomes small, and vanishes for equal radii. If, further, the lengths of the solenoids be equal also, $I_2 = I_3$, and we have only three of the integrals to evaluate, and only the first term I' in each of these.

For concentric, coaxial solenoids $d=0$, and consequently $I_1 - I_2 = I_4 - I_3$, so that only two integrals must be calculated.

On account of the number of auxiliary quantities involved, Nagaoka's formula should not be employed except when the various series formulas given in this section are all shown to be inadequate. It is, however, simpler to use Nagaoka's formula than the elliptic integral formula from which it is derived, or any other expression in incomplete integrals yet derived, even supposing Legendre's table of incomplete integrals to be available.

RUSSELL'S FORMULAS [40]

Russell's formula for coaxial solenoids in the notation of this paper is

$$\begin{aligned} M = 4\pi^2 a^2 n_1 n_2 \Bigg[R_1 \Bigg\{ 1 &- \frac{1}{2} q_2 k_1^2 - \frac{1}{2}\cdot\frac{1}{4} q_3 k_1^4 - \frac{1.1.3}{2.4.6} q_4 k_1^6 \\ &- \frac{1.1.3.5}{2.4.6.8} q_5 k_1^8 - \frac{1.1.3.5.7}{2.4.6.8.10} q_6 k_1^{10} - \cdots \Bigg\} \\ &- R_2 \Bigg\{ 1 - \frac{1}{2} q_2 k_2^2 - \frac{1}{2}\cdot\frac{1}{4} q_3 k_2^4 - \text{terms with above coefs.} \Bigg\} \Bigg] \end{aligned} \qquad [46]$$

where

$$R_1^2 = (A+a)^2 + (l_1 + l_2)^2 \qquad k_1^2 = \frac{4Aa}{R_1^2}$$

$$R_2^2 = (A+a)^2 + (l_1 - l_2)^2 \qquad k_2^2 = \frac{4Aa}{R_2^2}$$

[40] Alexander Russell, Phil. Mag., Apr. 1907, p. 420.

$$q_n = \frac{(A+a)^2}{4Aa} q_{n-1} - \frac{1}{n} \frac{1.3.5 \ldots . 2n-3}{2.4.6 \ldots . 2n-2} \frac{A}{a}$$

$$q_2 = \frac{(A+a)^2}{4Aa} - \frac{1}{2} \frac{1}{2} \frac{A}{a}$$

$$q_3 = \frac{(A+a)^2}{4Aa} q_2 - \frac{1}{3} \frac{1.3}{2.4} \frac{A}{a}$$

etc.

A and a are the radii of the outer and inner cylinders respectively, $2l_1$ and $2l_2$ their lengths, Fig. 20, and n_1, n_2 the number of turns of wire per cm in the two windings. This formula applies only when the inner coil is shorter than the outer. For two coils of equal length the second part of the above formula is not convergent, and hence it must be replaced by an expression in elliptic integrals. The formula thus becomes [41a] (equation 42 in Russell's paper)

$$M = 4\pi^2 a^2 n_1 n_2 \left[R_1 \left\{ 1 - \frac{1}{2} q_2 k_1^2 - \frac{1}{8} q_3 k_1^4 - \cdot \cdot \cdot \text{ as above} \right\} \right]$$

$$+ \frac{8\pi(A+a)}{3} n_1 n_2 [(A^2 + a^2)(F-E) - 2AaF] \qquad [47]$$

the modulus of the elliptic integrals being $k_2 = \frac{2\sqrt{Aa}}{A+a}$

This formula gives an accurate result for equal solenoids of considerable length, but Maxwell's formula (36) is just as accurate and much more convenient.

For short coils neither (46) nor (47) will apply, but for that case as well as other cases Russell's general formula may be used. As the latter is equivalent to (44) it is not here given.

MUTUAL INDUCTANCE OF A SHORT SECONDARY ON THE OUTSIDE OF A LONG PRIMARY

This is an important case in practice. Havelock[41] has shown that the mutual inductance of two such solenoids is the same as that of two coils with the same radii and lengths, but with the shorter coil inside. That is, the mutual inductance of a coil of length l and radius A outside of a coil of length x and radius a is the same as

[41] Phil. Mag., **15**, p. 343; 1908.

[41a] See correction to original formula Sci. Abs., **11**, No. 1847, 1908.

the mutual inductance of a coil of length x and radius A outside of a coil of length l and radius a.

The series formulas already given for the latter case may therefore be applied to the present case directly if the quantities l and x, or l_1 and l_2, be interchanged.

In Ròiti's formula we put, therefore, $l_1=\frac{l-x}{2}$ instead of $\frac{x-l}{2}$. The values of ρ_1 and ρ_2 are, however, unchanged and the formula may be used just as it stands.

Russell's formula being symmetrical in l_1 and l_2 requires no change whatever.

In Searle and Airey's formula we have to put

$$d=\sqrt{l^2+A^2}$$

$$L_2=3-4\frac{x^2}{a^2}$$

$$X_2=3-4\frac{l^2}{A^2}$$

$$L_4=\frac{5}{2}-10\frac{x^2}{a^2}+4\frac{x^4}{a^4}$$

etc.

Cohen's and Nagaoka's formulas apply without change as would be expected.

ROSA'S FORMULAS FOR SINGLE LAYER COILS OF EQUAL RADII

The mutual inductance of two coaxial single layer coils of equal radii is given by the following expression:

$$\frac{M}{N_1N_2}=M_0+\Delta M$$

where M_0 is the mutual inductance of the two parallel circles at the centers of the coils and ΔM is given by the following expression:[42]

[42] Rosa, this Bulletin, 2, p. 351; 1906.

$$\begin{aligned}
\varDelta M = 4\pi a \Bigg[& \frac{1}{24}\frac{b_1^2+b_2^2}{d^2}\left\{1+\frac{3}{8}\frac{d^2}{a^2}\left(\log\frac{8a}{d}-\frac{11}{6}\right)-\frac{45}{256}\frac{d^4}{a^4}\left(\log\frac{8a}{d}-\frac{97}{60}\right)\right. \\
& \left.+\frac{1050}{128^2}\frac{d^6}{a^6}\left(\log\frac{8a}{d}-\frac{54}{35}\right)-\frac{44100}{128^3}\frac{d^8}{a^8}\left(\log\frac{8a}{d}-\frac{3793}{2520}\right)+\cdots\right\} \\
& +\frac{\left(b_1^4+b_2^4+\frac{10}{3}b_1^2b_2^2\right)}{320d^4}\left\{1+\frac{1}{16}\frac{d^2}{a^2}-\frac{15}{256}\frac{d^4}{a^4}\left(\log\frac{8a}{d}-\frac{187}{60}\right)\right. \\
& \left.+\frac{2100}{128^2}\frac{d^6}{a^6}\left(\log\frac{8a}{d}-\frac{893}{420}\right)-\cdots\right\} \\
& +\frac{b_1^6+b_2^6+7(b_1^4b_2^2+b_1^2b_2^4)}{2688d^6}\left\{1+\frac{3}{160}\frac{d^2}{a^2}-\frac{3}{1024}\frac{d^4}{a^4}+\cdots\right\} \\
& +\frac{(b_1^8+b_2^8)+12(b_1^6b_2^2+b_1^2b_2^6)+\frac{126}{5}b_1^4b_2^4}{18432d^8}\left\{1+\frac{1}{112}\frac{d^2}{a^2}+\cdots\right\}\Bigg]
\end{aligned} \qquad [48]$$

For coils of equal breadth and equal radii (Fig. 21) $b_1 = b_2 = b$ and we may write the equation (48) as follows:

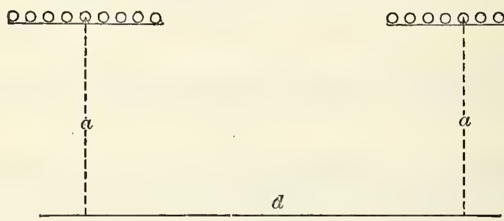

Fig. 21

$$\begin{aligned}
\varDelta M = 4\pi a \Bigg[& \frac{b^2}{12d^2}\left\{1+\frac{3}{8}\frac{d^2}{a^2}\left(\log\frac{8a}{d}-\frac{11}{6}\right)-\frac{45}{256}\frac{d^4}{a^4}\left(\log\frac{8a}{d}-\frac{97}{60}\right)\right. \\
& \left.+\frac{1050}{128^2}\frac{d^6}{a^6}\left(\log\frac{8a}{d}-\frac{54}{35}\right)-\frac{44100}{128^3}\frac{d^8}{a^8}\left(\log\frac{8a}{d}-\frac{3793}{2520}\right)+\cdots\right\} \\
& +\frac{1}{60}\frac{b^4}{d^4}\left\{1+\frac{1}{16}\frac{d^2}{a^2}-\frac{15}{256}\frac{d^4}{a^4}\left(\log\frac{8a}{d}-\frac{187}{60}\right)\right. \\
& \left.+\frac{2100}{128^2}\frac{d^6}{a^6}\left(\log\frac{8a}{d}-\frac{893}{420}\right)-\cdots\right\} \\
& +\frac{1}{168}\frac{b^6}{d^6}\left\{1+\frac{3}{160}\frac{d^2}{a^2}-\frac{3}{1024}\frac{d^4}{a^4}+\cdots\right\}+\frac{1}{360}\frac{b^8}{d^8}\left\{1+\frac{1}{112}\frac{d^2}{a^2}-\cdots\right\}\Bigg]
\end{aligned} \qquad [49]$$

This expression will give a very accurate value of ΔM for two coils not nearer together than their breadth if a is considerably greater than b, the breadth of the coil.

For coils which are not so near together the Rosa-Weinstein formula[43] may be used.

$$\Delta M = 4\pi a \sin\gamma \, [(F-E)\,P + EQ] \qquad [50]$$

where

$$P = \frac{\cos^2\gamma}{24d^2}[\alpha_1 - \alpha_3 - 3\alpha_3\cos^2\gamma + 8\alpha_3\cos^4\gamma]$$

$$Q = \frac{\sin^2\gamma}{24d^2}[\alpha_1 + 2\alpha_3 + 3\alpha_3\cos^2\gamma + 8\alpha_3\cos^4\gamma]$$

$$\alpha_1 = (b_1^2 + b_2^2) \qquad \alpha_3 = \frac{3(b_1^4 + b_2^4) + 10b_1^2 b_2^2}{80d^2}$$

$$\sin^2\gamma = \frac{4a^2}{4a^2+d^2} \qquad \cos^2\gamma = \frac{d^2}{4a^2+d^2}$$

and F and E are the complete elliptic integrals of the first and second kinds with modulus $k = \sin\gamma$.

When the coils have equal breadth $b_1 = b_2 = b$ and $\alpha_1 = b^2$, $\alpha_3 = \dfrac{b^4}{5d^2}$.

If the lengths of the coils are not very small in comparison with d a greater precision may be attained by adding to (50) the last two terms of (48) or (49) which depend on differentials of the sixth and eighth order.

MUTUAL INDUCTANCE BY MEANS OF SELF-INDUCTANCE FORMULA

The mutual inductance of two coils having the same radii and the same number of turns per unit of length may be calculated with great accuracy from a knowledge of several self-inductances.

If the two coils be designated as A and B and a coil C having the same radius and number of turns per unit length be imagined to exactly fill up the space between A and B, the self-inductance of coils A, B and C in series will be

$$L_{ABC} = L_A + L_B + L_C + 2M_{AC} + 2M_{BC} + 2M_{AB}$$

[43] This Bulletin, 2, p. 342; 1906.

Similarly the self-inductances of the coils A and C in series, and of B and C in series are given by the equations.

$$L_{AC} = L_A + L_C + 2M_{AC}$$

$$L_{BC} = L_B + L_C + 2M_{BC}$$

Eliminating M_{AC} and M_{BC} in the equation above we find

$$2M_{AB} = (L_{ABC} + L_C) - (L_{AC} + L_{BC}) \qquad [51]$$

The self-inductances may be calculated with all the accuracy desired by Lorenz's or Nagaoka's formulas. Formula (51) is of especial value in testing new formulas and in the case where the two coils are in contact. In the latter case the formula becomes

$$2M_{AB} = L_{AB} - (L_A + L_B) \qquad [52]$$

OTHER FORMULAS

Himstedt has given several formulas for different cases of coaxial solenoids. The first[44] is for the case of a short secondary on the outside of a long primary. The formula is very complicated, and the calculation tedious. The formulas of Ròiti and Searle and Airey may be used to much better advantage.

Himstedt's second expression is for the case of two coaxial solenoids not concentric, the distance between their mean planes having any value; the radius of one is supposed to be considerably smaller than the other. This also is a very complicated formula, involving second and fourth derivatives of expressions containing the elliptic integrals F and E. Gray's general equation is much simpler to calculate. This is not, however, an important case in practice, and we do not therefore give Himstedt's equation. Himstedt's third equation is general and applies to two coaxial solenoids of nearly equal or very different radii, which may or may not be concentric. This expression of Himstedt's consists of four terms, each of which is a somewhat complicated expression involving both complete and incomplete elliptic integrals. A less inconvenient general expression for M in elliptic integrals is given above (44).

Havelock[45] gave a formula for the mutual inductance of two coaxial, concentric solenoids, which resembles somewhat the formula

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[44] Wied. Annalen, **26**, p. 551; 1885.

[45] Phil. Mag., **15**, p. 342; 1908.

of Ròiti. It is, however, not so convergent as the latter, except when one coil is very short in comparison with the other.

After the present work had gone to press a valuable article appeared[45a] by Olshausen, in which the author derived a general absolute expression for the mutual inductance of two coaxial solenoids. Adopting the same nomenclature as in Nagaoka's formula (45), the integral I is in this case given by

$$I=\frac{(2Aam)^{\frac{3}{2}}}{2}\left[\left(\frac{g_2}{6}-\mathfrak{p}^2v\right)\omega_1+\mathfrak{p}v\cdot\eta_1+\frac{\mathfrak{p}'v}{2}\left\{\eta_1 v-\omega_1\frac{\sigma_1}{\sigma}(v)+n\pi i\right\}\right] \qquad [52a]$$

Here m is a parameter, which is to be arbitrarily assigned, and consequently (52a) may be put into various special forms depending on the value assumed for m. The integer n, which may be positive or negative, enters because of the many-valuedness of a logarithm, and is to be found from the equation defining v.

If we place $m=\left(\frac{2}{Aa}\right)^{\frac{1}{3}}$ and let $n=0$, the result is Nagaoka's equation (45). The author shows further that by expressing the quantities in (52a) in terms of the elliptic integrals of Legendre, Cohen's absolute formula (44) may be shown to be a special case of the general equation (52a).

As a third example, the author shows that the absolute formula of Kirchhoff, published for the first time by Coffin[45b] in a form subsequently shown by Cohen[45c] to be in error, is included in (52a), and the correct expression is given in Olshausen's equation (38).

Olshausen showed further that if the value $\frac{(A+a)^2+c^2}{2Aa}$ be assigned to m, the expressions for some of the auxiliary quantities become very simple. For the details of calculation as arranged by him we refer to the original article.

CHOICE OF FORMULAS

1. *Coaxial solenoids, not concentric.*—(a) For the general case, if the greatest precision is required, Nagaoka's absolute formula (45) should be used. Since, however, the mutual inductance of such

[45a] Phys. Rev., 31, p. 617; 1910.
[45b] This Bulletin, 2, p. 125; 1906.
[45c] This Bulletin, 3, p. 301; 1907.

coils will not in general be needed with extreme accuracy, it will usually be found sufficient to apply Gray's formula (40), taking the precaution to determine by a rough preliminary calculation, whether or not the terms of higher order will have an appreciable effect in the case at hand. For this purpose Table XVIII will be found of material aid.

If the convergence is not satisfactory, or if more than three or four terms must be calculated, it will be found advantageous to subdivide one or both of the coils, and to apply Gray's formula to the calculation of the mutual inductance of the several pairs of sections; for these the convergence will be more rapid. For example, if coil A be divided into two parts, C and D, and the coil B into sections E and F, then the mutual inductance of A or B will be given by the relation

$$M_{AB} = M_{CE} + M_{CF} + M_{DE} + M_{DF}$$

It may be stated as a general criterion for the rapid convergence of Gray's formula, that the distance between the coils should be great relatively to the radii, and that the coils should not be very long. With long coils it is necessary to carry the subdivision further than with short coils, with a corresponding increase in the number of terms to be calculated, but even then the labor will generally be much less than in using Nagaoka's formula.

If the coils be relatively far apart, and great precision is not desired, the formula of quadratures (23) may be adapted to this case, by making the radial dimension of the cross section of the coils in Fig. 4 equal to zero. We have then

$$M_1 = M_3 = M_5 = M_7 = M_0$$

and the formula of quadratures becomes

$$M = \frac{1}{6}(2M_0 + M_2 + M_4 + M_6 + M_8)$$

It is, therefore, only necessary to calculate, by an appropriate formula or formulas, the mutual inductances of the five pairs of circles, and to take the weighted mean indicated. This formula is more accurate, the shorter the axial lengths of the solenoids in comparison with their distance apart, and the process of subdivision above described will be, in general, necessary. Gray's formula is, however, to be preferred.

(*b*) An important case in practice is that of solenoids of *equal radii*. If the coils be in contact or very near together the formulas (52) or (51), respectively, should be employed.

If the solenoids be separated so that the distance between their medial planes is greater than the axial length of either, the mutual inductance may be calculated from the mutual inductance of the two circles at their centers, a correction being applied to take account of the lengths of the coils. For this purpose formula (48) should be used for coils relatively near together and (50) for coils farther apart. The corresponding formulas, for coils of *equal radii* and *equal length* are (49) and (50).

2. *Coaxial, concentric solenoids of equal length*.—If the solenoids be long relatively to their radii, Havelock's formula (38) will be found to be very accurate. Maxwell's formula (36), however, is applicable to both long and short solenoids, provided the radii are not too nearly equal, and should be given the preference, using Havelock's, when desired, as a check on the result. It may be necessary in rare cases to use the absolute formulas of Nagaoka or Cohen. One should also bear in mind that Ròiti's and Searle and Airey's formulas also hold for equal length solenoids, and may be used in checking the results.

3. *Coaxial, concentric solenoids—Inner coil the shorter*.—For relatively long coils Ròiti's formula (39) will give very accurate values, whatever the length of the inner solenoid, provided the radius of the inner coil is not closely equal to that of the outer. Ròiti's formula is also applicable to short solenoids in case the inner radius is considerably smaller than the outer. For short solenoids, however, Searle and Airey's formula (43) is preferable, and gives a very rapidly converging value unless the inner radius be nearly equal to the outer. Russell's formula (46) is most convergent for long solenoids, of which the inner one is a good deal shorter than the outer one.

In those cases for which none of the above formulas converge rapidly, and great precision is desired, Nagaoka's or Cohen's absolute formula should be used.

4. *Coaxial, concentric solenoids—Outer coil the shorter*.—The formulas of the preceding section are to be used interchanging x and l, or l_1 and l_2 as the case may be. The formulas of Ròiti,

Russell, Cohen, and Nagaoka are unchanged in form; Searle and Airey's is slightly changed as regards the coefficients L and X.

Usually, it will be found that more than one formula will apply to a given case. The advantage of such a check can not be overestimated.

For illustrations and tests of the above formulas, see examples 34–47.

In taking the dimensions of coils where an accurate value of the mutual inductance is sought it should be borne in mind that the above formulas have been derived on the supposition that the current is uniformly distributed over the length of the coaxial solenoids; in other words, these formulas are all current-sheet formulas. Hence, for coils made up of many turns of wire we must meet the conditions imposed by current-sheet formulas. In calculating self-inductances, this requires an accurate determination of the size of the wire and of the distance between the axes of successive wires, from which we can calculate two correction terms to be combined with the value of L given by the current-sheet formulas.[46]

In the case of mutual inductances, however, there are no correction terms to calculate; but we must take the dimensions of the current sheets that are equivalent to the coils of wire; that is, the radius of each coil will be the mean distance to the center of the wire, and the length of each will be the over-all length, including the insulation, when the coil is wound of insulated wire in contact, or the length from center to center of a winding of $n+1$ turns, where n is the whole number of turns used.[47] It is also very important that the winding on both coils shall be uniform,[48] and that the leads shall be brought out so that there shall be no mutual inductance due to them.

The mutual inductance will of course be different at high frequencies from its value at low frequencies. We assume, however, that for all purposes for which an extremely acturate mutual inductance is desired the frequency of the current would be low, say,

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[46] Rosa, this Bulletin, **2**, p. 181; 1906.

[47] Rosa, this Bulletin, **2**, p. 161, 1906; and **3**, p. 1; 1907.

[48] Searle and Airey, Electrician (London), **56**, p. 318; 1905.

not more than a few hundred per second. If the value at very high frequency is desired the coil should be wound with stranded wire, each strand of which is separately insulated.

EXAMPLES ILLUSTRATING THE FORMULAS FOR THE MUTUAL INDUCTANCE OF COAXIAL SOLENOIDS

EXAMPLE 34. MAXWELL'S FORMULA (36) AND COHEN'S (44)

Two solenoids, Fig. 22, of equal length, 200 cm, each wound with a single layer coil.

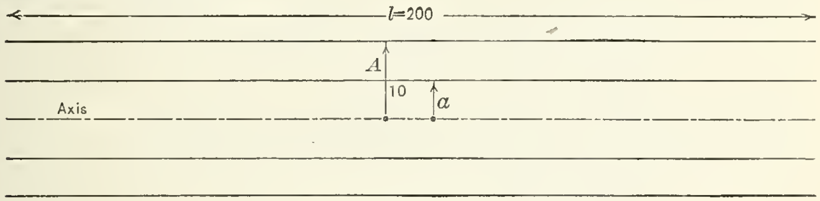

Fig. 22

$$A = 10 = \text{radius of outer.}$$
$$a = \ \ 5 = \text{radius of inner.}$$

Substituting in (36) for α we have the following:

$$\alpha = 0.487508 - \frac{1}{16}\frac{a^2}{A^2}(0.999875) - \frac{1}{64}\frac{a^4}{A^4}(0.500001) - \frac{35}{2048}\frac{a^6}{A^6}\left(\frac{1}{7}\right)$$
$$= 0.487508 - 0.015623 - 0.000488 - 0.000038$$
$$= 0.471359$$
$$\therefore M = 4\pi^2 a^2 n^2 (200 - 9.42718)$$
$$M = 19057.28\pi^2 n^2$$

If $n = 10$ turns per cm, $M = \dfrac{100\ \pi^2 \times 19057.28}{10^9}$ henry

$$= 0.01880878 \text{ henry.}$$

The effect of the next term in the series for α beyond those calculated is to raise the value of M by only one part in five million.

By the approximate formula of Maxwell (37)

$$2\alpha = 1 - \frac{1}{8 \times 4} - \frac{1}{64 \times 16} - \frac{1}{1024 \times 64} - \ \cdot \ \cdot\cdot\cdot$$
$$= 0.96773$$
$$\therefore M = 0.018784 \text{ henry.}$$

This example by Heaviside's extension of Maxwell's formula (see p. 55) has exactly the same value of M; that is, the added terms do not amount to as much as a millionth of a henry in this particular case.

To show that the result by Maxwell's formula (36) is very accurate for this case we may now calculate M by Cohen's absolute formula:

$$M = 4\pi n^2 (V - V_1)$$

Substituting in (44) for V we have the following terms:

$$\begin{aligned} V &= 7863.79 + 4200532.04 - 4169106.25 - 23561.95 \\ &= 15727.63 \\ V_1 &= 1392.18 - 632.16 = 760.02 \\ \therefore M &= 4\pi n^2 \,(15727.63 - 760.02) = 4\pi n^2 \,(14967.61) \\ &= 19057.36 \pi^2 n^2 \\ M &= 0.01880886 \text{ henry.} \end{aligned}$$

This agrees with the result by Maxwell's formula to within five parts in a million, the value by Maxwell's formula being more nearly correct, as is shown in the next example.

The example by Cohen's formula illustrates the disadvantage of that formula for numerical calculations. Aside from the fact that it is complicated, and involves the use of both complete and incomplete elliptic integrals, the value of M depends on the difference between very large positive and negative terms, so that in order to get a value of M correct to one part in one hundred thousand it is necessary in the above example to calculate the large terms to one part in twenty-five million. As a means of testing other formulas, however, this absolute formula is of great value.

EXAMPLE 35. HAVELOCK'S FORMULA (38)

We will take the same problem as in the preceding example:

$$a = 5 \qquad A = 10 \qquad l = 200$$

$$\frac{1}{2} - \frac{1}{16}\frac{a^2}{A^2} = 0.484375$$

$$-\frac{1}{128}\frac{a^4}{A^4} = -0.000488$$

$$-\frac{5}{2048}\frac{a^6}{A^6} = -0.000038$$

$$-\frac{1}{4}\frac{A}{l} = -0.012500$$

$$\frac{1}{16}\left(1+\frac{a^2}{A^2}\right)\frac{A^3}{l^3} = +0.000010$$

$$\text{Sum} = \beta = 0.471359$$

which is exactly the same as the value of α found by Maxwell's formula in the preceding example. The value of the mutual inductance agrees, therefore, exactly to seven significant figures with the value given by Maxwell's formula. For this example, accordingly, we see that Maxwell's and Havelock's formulas give a more accurate value than Cohen's formula, unless the quantities in the latter are carried out to a greater number of places of decimals. This was pointed out by Havelock.[49]

EXAMPLE 36. MAXWELL'S FORMULA (36). FOR EQUAL SHORT SOLENOIDS

$$a=5 \qquad A=10 \qquad l=2$$

$$r=\sqrt{104} \qquad \frac{A}{r}=0.9805808 \qquad \frac{a}{A}=\frac{1}{2}$$

$$\frac{A-r+l}{2A} = 0.09009805$$

$$-\frac{1}{16}\frac{a^2}{A^2}\left(1-\frac{A^3}{r^3}\right) = -\ 0.00089271$$

$$-\frac{1}{64}\frac{a^4}{A^4}\left(\frac{1}{2}+2\frac{A^5}{r^5}-\frac{5}{2}\frac{A^7}{r^7}\right) = -\ 0.00013073$$

$$-\frac{35}{2048}(0.080378)\frac{a^6}{A^6} = -\ 0.00002146$$

$$-\frac{63}{2.128^2}(0.5079)\frac{a^8}{A^8} = -\ 0.00000381$$

$$-\frac{231}{512^2}(0.788)\frac{a^{10}}{A^{10}} = -\ 0.00000068$$

$$-\frac{429}{2.1024^2}(2.43)\frac{a^{12}}{A^{12}} = -\ 0.00000012$$

$$\text{Sum} = \alpha = 0.08904854$$

[49] Phil. Mag., **15**, p. 341.

55376°—16——6

$$2A\alpha = \quad 1.7809708$$
$$l - 2A\alpha = \quad 0.2190292$$
$$\therefore \frac{M}{4\pi n_1 n_2} = \quad 17.20251$$

The formula is not so favorable in this case as for long coils, since the quantity $2A\alpha$ is nearly equal to l. Further, the quantities involved in the parentheses are rather large, although their sum is in only one case greater than unity. There is, however, no difficulty in obtaining these factors with all the accuracy required. We have carried out the calculation with this formula further than would in practice be desired, in order to test the formula. We find that to get the same order of accuracy by Searle and Airey's formula terms including the product $X_{12}L_{14}$ must be calculated. The result found was

$$\frac{M}{4\pi n_1 n_2} = 17.20252$$

or only one part in two million different. We have also calculated the mutual inductance of these coils by Nagaoka's formula and obtained a value not very different, but this is a very unfavorable case for this formula, no great accuracy being obtainable using seven-place logarithms.

EXAMPLE 37. RÒITI'S FORMULA (39) COMPARED WITH SEARLE AND AIREY'S (43)

We will now calculate the example, Fig. 23 (originally given by Searle and Airey[50]), by Ròiti's formula, and also by the formula of Searle and Airey.

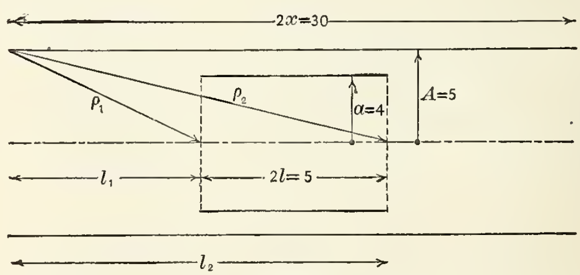

Fig. 23

[50] Electrician (London), 56, p. 319; 1905.

30 cm = length of outer solenoid.
5 " = " " inner "
$A =$ 5 " = radius " outer "
$a =$ 4 " = " " inner "

$$N_1 = 300 \text{ turns} \therefore n_1 = \frac{300}{30} = 10 \text{ per cm}$$

$$N_2 = 200 \quad " \quad n_2 = \frac{200}{5} = 40 \text{ per cm}$$

$$l_1 = 12.5 \qquad \rho_1 = \sqrt{12.5^2 + 25} = 13.462912$$

$$l_2 = 17.5 \qquad \rho_2 = \sqrt{17.5^2 + 25} = 18.200275$$

$$\therefore \rho_2 - \rho_1 = 4.737363$$

$$\rho_1 + \rho_2 = 31.663187$$

It is more accurate to calculate $(\rho_2 - \rho_1)$ by the formula

$$\rho_2 - \rho_1 = \frac{xl}{\rho_1 + \rho_2}.$$

This gives $(\rho_2 - \rho_1) = 4.7373620$, which value will be used in the calculation of M.

$$\rho_2 - \rho_1 = \quad 4.7373620$$

$$\frac{A^2a^2}{8}\left(\frac{1}{\rho_1^3} - \frac{1}{\rho_2^3}\right) = + \ .0121975$$

$$-\frac{A^4a^2}{16}\left(\frac{1}{\rho_1^5} - \frac{1}{\rho_2^5}\right) = - \ .0007041$$

$$\frac{5}{64}A^4a^4\left(1 + \frac{1}{2}\frac{a^2}{A^2}\right)\left(\frac{1}{\rho_1^7} - \frac{1}{\rho_2^7}\right) = + \ .0001808$$

$$-\frac{35A^4a^6}{256}\left(1 + \frac{1}{5}\frac{a^2}{A^2}\right)\left(\frac{1}{\rho_1^9} - \frac{1}{\rho_2^9}\right) = - \ .0000254$$

$$+\frac{105A^6a^6}{1024}\left(1 + \frac{9}{5}\frac{a^2}{A^2} + \frac{1}{5}\frac{a^4}{A^4}\right)\left(\frac{1}{\rho_1^{11}} - \frac{1}{\rho_2^{11}}\right) = + \ .0000054$$

$$-\frac{693A^6a^8}{2048}\left(1 + \frac{2}{3}\frac{a^2}{A^2} + \frac{1}{21}\frac{a^4}{A^4}\right)\left(\frac{1}{\rho_1^{13}} - \frac{1}{\rho_2^{13}}\right) = - \ .0000010$$

$$\frac{3003A^8a^8}{16384}\left(1 + 4\frac{a^2}{A^2} + \frac{10}{7}\frac{a^4}{A^4} + \frac{1}{14}\frac{a^6}{A^6}\right)\left(\frac{1}{\rho_1^{15}} - \frac{1}{\rho_2^{15}}\right) = + \ .0000002$$

$$\text{Sum} = \quad 4.7490149$$

$$4\pi^2 a^2 n_1 n_2 = 25600\ \pi^2$$

$$\therefore\ M = \frac{25600\ \pi^2 \times 4.7490149}{10^9}\ \text{henry}$$

$$\text{or } M = 0.0011998950\ \text{henry.}$$

The sum of the next two terms in the series is equal to about one part in ten million. The value of the mutual inductance is therefore given with great precision by this formula. If the inner radius had been relatively smaller, the convergence would have been more rapid. We have, however, carried the computation much further than would in practice be necessary.

Calculating the same problem by Searle and Airey's formula we have

$$2x = 30 \qquad 2l = 5 \qquad A = 5 \qquad a = 4$$
$$N_1 = 300 \qquad N_2 = 200$$

$$d = \sqrt{250} \qquad \frac{A^2 a^2}{d^4} = \frac{4}{625}$$

$$\begin{array}{ll} X_2 = -\ 33.00 & L_2 = \ \ 1.43750 \\ X_4 = \ \ 236.5 & L_4 = -0.7959 \\ X_6 = -1370 & L_6 = -1.71 \\ X_8 = \ \ 4869 & L_8 = -0.72 \end{array}$$

$$\begin{array}{rl} 1 + \dfrac{A^2 a^2}{d^4} \cdot \dfrac{L_2}{8} = & 1.0011500 \\ \dfrac{A^4 a^4}{d^8}\ \dfrac{L_4 X_2}{32} = & 0.0000344 \\ \dfrac{A^6 a^6}{d^{12}}\ \dfrac{L_6 X_4}{32} = & -0.0000033 \\ \text{Sum} = & 1.0011811 \\ \dfrac{2\pi^2 a^2 N_1 N_2}{d} = & 1198480.5 \\ \therefore\ M = & 0.0011998957\ \text{henry.} \end{array}$$

The terms neglected are less than one part in ten million. The value of the mutual inductance found is only six parts in ten million greater than that found by Ròiti's formula, and for this problem the convergence of Searle and Airey's formula is the more satisfactory.

The same problem by Russell's formula (46) (extended to include six terms in each part of the formula) gives

$$M = 0.00119989 \text{ henry.}$$

Of the three formulas Searle and Airey's is for this case the most convergent, and Russell's the least convergent. If the ratio of the radii was still more nearly equal to unity, Searle and Airey's formula would still be satisfactory; the convergence of Ròiti's formula would, however, become poorer.

If in the above problem the length $2l$ of the inner coil be increased without changing the radii, the quantities L_{2n} in Searle and Airey's formula would become rapidly larger, and the convergence would become poorer. Ròiti's formula also becomes less satisfactory as l is increased. For $2l = 24$, however, Searle and Airey's formula will still give the correct result to about one part in 100000, but Ròiti's formula in this case converges very slowly. On the other hand, if the radius of the inner coil were smaller in the latter case, Ròiti's formula could be used, but Searle and Airey's would not converge rapidly enough. This is shown in the next example.

EXAMPLE 38. RÒITI'S FORMULA COMPARED WITH SEARLE AND AIREY'S FORMULA. COILS OF NEARLY EQUAL LENGTH

Length of outer solenoid = 30 cm

" inner " = 24

$a = 2 \qquad A = 5 \qquad n_1 = 10 \qquad n_2 = 40$

In Ròiti's formula:

$$\begin{aligned}
\rho_2 - \rho_1 &= \ 21.628108 \\
\text{2d term} &= +\ 0.062447 \\
\text{3d} \quad `` &= -\ 0.003707 \\
\text{4th} \quad `` &= +\ 0.003682 \\
\text{5th} \quad `` &= -\ 0.000724 \\
\text{6th} \quad `` &= +\ 0.000501 \\
\text{7th} \quad `` &= -\ 0.000170 \\
\text{8th} \quad `` &= +\ 0.000159 \\
\text{Sum} &= \ 21.690296 \\
4\pi^2 a^2 n_1 n_2 &= \ 6400\pi^2 \\
\therefore M &= \ 0.00137008 \text{ henry.}
\end{aligned}$$

In Searle and Airey's formula

$$X_2 = -\ 33 \qquad L_2 = -\ 141$$
$$X_4 = 236.5 \qquad L_4 = 4826.5$$
$$X_6 = -1370.3 \qquad L_6 = -160036$$
$$X_8 = 4869 \qquad L_8 = 5.019 \times 10^7$$
$$X_{10} = 15746 \qquad L_{10} = -1.571 \times 10^8$$
$$L_{12} = 4.483 \times 10^9$$

$$d^2 = 250$$

$$\begin{aligned} & \ 1.000000 \\ \text{2d term} &= -0.028200 \\ \text{3d} \quad \text{“} &= -0.012742 \\ \text{4th} \quad \text{“} &= -0.004845 \\ \text{5th} \quad \text{“} &= -0.001409 \\ \text{6th} \quad \text{“} &= -0.000251 \\ \text{7th} \quad \text{“} &= +0.000037 \\ \text{Sum} &= 0.952590 \end{aligned}$$

$$\frac{2\pi^2 a^2 N_1 N_2}{d} = \frac{2304000\pi^2}{d}$$

$$\therefore M = 0.00136999 \text{ henry.}$$

In this case we see that the higher order terms in Ròiti's formula arrange themselves in pairs of nearly equal values with opposite signs. The convergence is, therefore, better than appears at first sight, and the terms here neglected do not amount to more than one part in a million in M. Searle and Airey's formula does not converge so rapidly, the eighth and still higher order terms being appreciable. If the length of the inner solenoid were made still greater the L coefficients would become even larger than they are here, and the convergence would become unsatisfactory.

EXAMPLE 39. RÒITI'S AND SEARLE AND AIREY'S FORMULAS IN THE CASE OF SHORT COILS

Length of the outer solenoid = 5 cm
" " " inner " = 2

$$a = 2 \qquad A = 10$$

In Ròiti's formula:

$$\rho_1 = 10.111873$$
$$\rho_2 = 10.594808$$

which used in the formula $\frac{xl}{\rho_1+\rho_2}$ give a more accurate value of $\rho_2-\rho_1$, viz:

$$
\begin{aligned}
\rho_2-\rho_1 &= 0.4829359 \\
\text{2d term} &= 0.0063160 \\
\text{3d ''} &= -0.0001968 \\
\text{4th ''} &= 0.0003286 \\
\text{5th ''} &= -0.0000274 \\
\text{6th ''} &= 0.0000250 \\
\text{7th ''} &= -0.0000035 \\
\text{8th ''} &= \underline{0.0000023} \\
\text{Sum} &= 0.4893801 \\
4\pi^2a^2 &= 16\pi^2 \\
\therefore \frac{M}{n_1n_2} &= 77.27981
\end{aligned}
$$

In Searle and Airey's formula

$$
\begin{aligned}
X_2 &= 2.75 & L_2 &= 2 \\
X_4 &= \frac{121}{64} & L_4 &= \frac{1}{4} \\
X_6 &= \frac{1203}{1024} & L_6 &= -\frac{15}{16} \\
X_8 &= \frac{9265}{16384} & L_8 &= -\frac{77}{64} \\
d^2 &= 106.25 & L_{10} &= -\frac{9}{16}
\end{aligned}
$$

$$
\begin{aligned}
& 1.0000000 \\
\text{1st term} &= 0.0088581 \\
\text{2d ''} &= 0.0000270 \\
\text{3d ''} &= -0.0000025 \\
\text{4th ''} &= -\underline{0.0000001} \\
\text{Sum} &= 1.0088825 \\
\frac{2\pi^2a^2N_1N_2}{d} &= \frac{80\pi^2n_1n_2}{d} \\
\therefore \frac{M}{n_1n_2} &= 77.27980
\end{aligned}
$$

The neglected terms in Searle and Airey's forumula are entirely inappreciable. The convergence of Ròiti's formula is not quite so

good. The sum of the next two terms is such as to reduce M by three units in the last place, but the following terms do not decrease very rapidly. We may evidently regard the use of either formula as entirely justified in the problem. If the radius of the outer coil had been only one-half as great, the lengths of the two coils and the radius of the inner remaining unchanged, the value found by Ròiti's formula would be in error by more than one part in ten thousand; the convergence of Searle and Airey's would, however, be satisfactory in this case. This formula would, on the other hand, be less convergent when the length of the inner coil is nearly as great as that of the outer coil. In general, it will be found that these two formulas between them cover sufficiently well the cases which may arise in practice.

EXAMPLE 40. GRAY'S FORMULA (41) COMPARED WITH RÒITI'S (39)

Let the winding be 20 turns per cm on each coil (Fig. 24), $n_1 = n_2 = 20$.

$$A = 25 \text{ cm} \qquad N_1 = n_1 A\sqrt{3}$$

$$a = 10 \text{ cm} \qquad N_2 = n_2 a\sqrt{3}$$

$$\therefore N_1 N_2 = 3n_1 n_2 A a$$

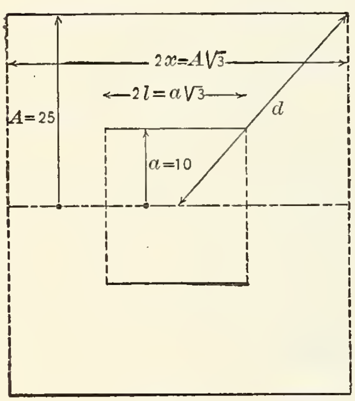

Fig. 24

$$d = \sqrt{x^2 + A^2} = \frac{A}{2}\sqrt{7}$$

$$\therefore \quad M = \frac{2\pi^2 a^2 N_1 N_2}{d} = 4\pi^2 a^2 n_1 n_2 \left[\frac{3a}{\sqrt{7}}\right]$$

$$M = .0179057 \text{ henry.}$$

We have also calculated the mutual inductance for these coils by Ròiti's formula (39).

The value is, $M = .0179058$, which is practically identical with the value by Gray's formula.

When $A = 25$ cm and $a = 10$ cm, $N_1 = 20A\sqrt{3} = 866.025$ and $N_2 = 20a\sqrt{3} = 346.4$. As there must be an integral number of turns, let us suppose the winding is exactly 20 turns per cm on each coil and the lengths therefore 43.3 cm and 17.3 cm, respectively. Then $d = \sqrt{x^2 + A^2} = \sqrt{625 + \left(\frac{43.3}{2}\right)^2} = 33.0715$ cm. This does not exactly

conform to the condition imposed in deriving the simple formula (41) of Gray used above. Hence (41) will not be as exact with these slightly altered dimensions, and we must calculate at least one correction term to get an accurate value of M.

By Gray's formula (41), $M=\dfrac{2\pi^2 100\times 866\times 346}{33.0715\times 10^9}=.0178842$ henry.

The first correction term in (43) increases this value to .0178854 henry.

We will now calculate the mutual inductance of these coils by Ròiti's formula (39):

$A=25 \quad 2x=43.3 \quad l_1=13.0$ cm

$a=10 \quad 2l=17.3 \quad l_2=30.3$ cm

$\rho_1=$	28.17800
$\rho_2=$	39.28218
$\rho_2-\rho_1=$	11.10418
2nd term $=+$	.22030
3rd $=-$	.01781
4th $=+$	.01952
5th $=+$	.00156
6th $=-$	.00453
7th $=+$	.00274
Sum $=$	11.32596

$$M=\frac{4\pi^2 a^2 n_1 n_2\times 11.32596}{10^9} \text{ henry,}$$

$=.0178853$ henry.

This differs from the result by Gray's formula by only 1 part in 178000.

EXAMPLE 41. GRAY'S FORMULA (40) COMPARED WITH NAGAOKA'S FORMULA (45)

We will next consider a practical problem suggested by Prof. Nasmyth.

$2x=20.55 \quad A=6.44 \quad N_1=15$ turns.

$2l=27.38 \quad a=4.435 \quad N_2=75$ "

The distance between the adjacent ends of the two solenoids was 7.2 cm. From this we find

$n_1=0.7296$ turns per cm

$n_2=2.737$ " " "

$x_1=20.89$

$x_2=41.44$

$k_1K_1=0.042937$

$k_3K_3=0.018274$

$k_5K_5=0.005193$

$k_7K_7=0.001423$

$k_9K_9=0.000116$

Sum $=0.067943$

$\therefore$ M $=1092.3$ cm.

It is evident that the convergence is not rapid enough to give a very precise value for the mutual inductance. We next divide the longer coil S into two sections C and D, such that C has 37 turns and D has 38 turns, C being the section nearer the other coil R. The axial lengths of these sections are, respectively, 13.51 and 13.87 cm. It would be just as well, if not better, to divide the coil into two equal sections of 37½ turns each. The division chosen was for greater convenience in the solution of the same problem by the method of quadratures. (Example 42.)

We now proceed to find M_{RC} and M_{RD}. The L coefficients are much smaller than before on account of the ratio $\frac{l}{a}$ being now smaller than was previously the case, and the convergence is much more satisfactory. These coefficients would be still smaller if we had divided coil R instead of S into two sections, measuring the x's from the center of R instead of using S for the reference coil as is done here. This advantage would, nevertheless, be in large measure offset by the smaller values of the distances r_1 and r_2.

We find for M_{RC}

$$\begin{aligned} k_1K_1 &= 0.048894 \\ k_3K_3 &= 0.006520 \\ k_5K_5 &= 0.000051 \\ \hline \text{Sum} &= 0.055465 \\ \therefore M_{RC} &= \quad 891.7 \text{ cm} \end{aligned}$$

and for M_{RD}

$$\begin{aligned} k_1K_1 &= 0.011549 \\ k_3K_3 &= 0.000613 \\ k_5K_5 &= 0.000004 \\ \hline \text{Sum} &= 0.012166 \\ \therefore M_{RD} &= \quad 195.6 \text{ cm.} \end{aligned}$$

Consequently $M = M_{RC} + M_{RD} = 1087.3$ cm.

To test the correctness of this value, the coil R was divided into two sections A and B (B being the section nearer to S), and the four mutual inductances between these sections and the two sections C and D of the coil S were calculated.

	M_{AC}	M_{BC}	M_{AD}	M_{BD}
$k_1K_1=$	0.012024	0.036869	0.003893	0.007657
$k_3K_3=$	0.000863	0.005654	0.000141	0.000471
$k_5K_5=$	0.000004	0.000047	0.000001	0.000004
	0.012891	0.042570	0.004035	0.008132
	$M_{AC}=207.2$	$M_{BC}=684.4$	$M_{AD}=64.9$	$M_{BD}=130.7$

$$M = \text{sum} = 1087.2 \text{ cm.}$$

The only component for which the convergence was not entirely satisfactory was M_{BC}. Here the sections are relatively near together and the coefficients L and X are not very favorable. Accordingly M_{BC} was calculated by two other methods (a) by dividing B into two sections, H and J, and by calculating M_{HC} and M_{JC}, (b) by dividing C into two sections, F and G, and by calculating M_{BF} and M_{BG}. The first procedure, on account of the relatively smaller values of r_1 and r_2, did not give a satisfactory degree of convergence. The latter, however, is better, the values found being

$$M_{BC} = 463.8 + 220.0 = 683.8$$

Using this value instead of the above the value of M is 1086.6 cm.

As the final check we have calculated the mutual inductance by Nagaoka's formula (45). The entire calculation has to be carried through for four different values of c, viz, 55.13, 34.58, 27.75, and 7.20. The corresponding values of I are

$$I_1 = 60.041802$$
$$I_2 = 38.047638$$
$$I_3 = 30.811676$$
$$I_4 = 10.333503$$

and $(I_1+I_4)-(I_2+I_3) = 70.375305 - 68.859314$

$$= 1.515991$$

$$\therefore M = 4\pi n_1 n_2 Aa(1.515991) = 1086.55 \text{ cm.}$$

An inspection of the various details of the calculation shows that the last figure may be in error by several units, although the utmost precision of which the seven-place logarithms are capable was striven

for. Of course by carrying the computation of the various quantities to a still greater number of decimal places, the accuracy of the result would be enhanced. Similarly the component values of the mutual inductance by Gray's formula would have been more accurate if we had not stopped with k_5 K_5. Since the dimensions of such coils would not ordinarily be obtained with greater precision than the accuracy here attained by Gray's formula, it is evident that the latter is for such cases much to be preferred to Nagaoka's formula, and the same would be true if the number of components were considerably increased. Nagaoka's formula has nevertheless the advantage in checking other formulas.

EXAMPLE 42. FORMULA OF QUADRATURES

The problem treated in the preceding example may also be solved by the formula of quadratures, using formula (8) for the calculation of the mutual inductance of the various pairs of circles. In general the method is not so accurate as that in the preceding example, and no time is saved. Only the results are here given, together with those by Gray's formula for comparison.

Single coils	Two sections	Four sections		
$M = 964.1$	$M_{RC} = 848.1$	$M_{AC} = 205.7$		$M_{BF} = 504.5$
	$M_{RD} = 193.7$	$M_{BC} = 669.5$		$M_{BG} = 182.2$
	$M = 1041.8$	$M_{RC} =$	875.2	$M_{BC} = 686.7$
		$M_{AD} = 66.5$		
		$M_{BD} = 130.4$		
		$M_{RD} =$	196.9	
			$M = 1072.1$	

Using the value of M_{BC} in the last column, $M = 1089.3$.

By Gray's formula:

Single coils	Two sections	Four sections		
$M = 1092.3$	$M_{RC} = 891.7$	$M_{AC} = 207.2$		$M_{CH} = 463.8$
	$M_{RD} = 195.6$	$M_{BC} = 684.4$		$M_{CJ} = 220.0$
	$M = 1087.3$	$M_{RC} =$	891.6	$M_{BC} = 683.8$
		$M_{AD} = 64.9$		
		$M_{BD} = 130.7$		
		$M_{RD} =$	195.6	
			$M = 1087.2$	

Using the value of M_{BC} in the last column, $M = 1086.6$.

EXAMPLE 43. MUTUAL INDUCTANCE OF CONCENTRIC COAXIAL SOLENOIDS BY NAGAOKA'S FORMULA (45)

$$2x = 200 \text{ cm}$$
$$2l = 20$$
$$A = 15$$
$$a = 10$$

This pair of coils was used by Cohen[51] in testing his absolute formula. He gave as the result $M = 4\pi n_1 n_2$ (6213.4). The same problem was worked out by Nagaoka[52] as an illustration of the use of his formula, the value of M found by him being $M = 4\pi n_1 n_2$ (6213.51). We have repeated his calculation, which was given only in abbreviated form, and agree substantially with his result, the value found being $M = 4\pi n_1 n_2$ (6213.52). Using seven-place logarithms it is very difficult to be sure of the last place of decimals given here. On the other hand, we find with Ròiti's formula, only three terms being necessary $M = 4\pi n_1 n_2$ (6213.509), and the same number of terms in Searle and Airey's formula give $M = 4\pi n_1 n_2$ (6213.510) with no uncertainty greater than one unit in the last place given. Olshausen found for the same coils the values $4\pi n_1 n_2$ (6213.77), and $4\pi n_1 n_2$(6213.63), using two methods of calculation in his formulas (21) and (61) and with $m = \frac{(A+a)^2 + c^2}{2aA}$. By the Kirchhoff formula he found $4\pi n_1 n_2$(6212.9). This is, however, an unfavorable case for this formula, since the angles φ and θ, on which the incomplete integrals $E(\varphi, k')$ and $F(\varphi, k')$ depend, are too near 90° to allow of accurate interpolation in Legendre's tables.

These differing results by the various absolute formulas, which arise from the fact, that in all of them the auxiliary quantities must be calculated with a considerably greater degree of accuracy than that desired in the result, serve to emphasize the advantage of the series formulas. In the great majority of practical cases the values, found by the use of series formulas, are not only obtained with a much smaller expenditure of labor, but are more accurate than when an absolute formula is used.

[51] This Bulletin, 3, p. 8; 1907.

[52] Jour. Tokyo, Math. Phys. Soc. (2), 4, p. 284; 1908.

We reproduce below the principal results in the calculation of this problem by Nagaoka's formula.

$$A+a=25 \qquad A-a=5 \qquad A^2-a^2=125$$
$$(A+a)^2=625 \qquad (A-a)^2=25$$

The argument of $I_1=-I_3$ is $c=110$; that of $I_2=-I_4$ is $c=90$.

	C=110	C=90
$\log \alpha=$	$\bar{1}.3749796$	$\bar{1}.3749796$
$\log 2Aa\alpha=$	1.8521009	1.8521009
$P_1=\mathfrak{p}v-e_1=$	-170.09225	-113.86339
$P_2=\mathfrak{p}v-e_2=$	$+\ 0.3514302$	$+\ 0.3514302$
$P_3=\mathfrak{p}v-e_3=$	$+\ 8.7857551$	$+\ 8.7857551$
$\mathfrak{p}v=\frac{1}{3}(P_1+P_2+P_3)=$	$-\ 53.651687$	$-\ 34.908733$
$e_2-e_3=$	8.4343249	8.4343249
$e_1-e_3=$	178.87801	122.64916
$e_1-e_2=$	170.44368	114.21483
$k'=\sqrt{\frac{e_1-e_2}{e_1-e_3}}=$	0.9761398	0.9650037
$1+\sqrt{k'}=$	1.9879979	1.9823461
$\log k^2=$	$\bar{2}.6734934$	$\bar{2}.8373857$
$q=$	0.0030186570	0.0044528010
$\log \omega_1$ (1st equat.)$=$	$\bar{1}.0750696$	$\bar{1}.1594886$
" (2d ")$=$	$\bar{1}.0750696$	$\bar{1}.1594886$
" (3d ")$=$	$\bar{1}.0750696$	$\bar{1}.1594887$
$P_2+P_3=$	9.1371853	9.1371853
$P_1+2P_2=$	-169.38939	-113.16053
$P_1+2P_3=$	-152.52074	$-\ 96.29188$
$\frac{P_1}{2}(P_2+P_3)=$	$-\ 777.08214$	-520.19542
$\frac{(e_2-e_3)}{6}(P_1+2P_2)=$	$-\ 238.11479$	-159.07205
$-$Sum$=$	$+1015.19693$	679.26747
$\times\omega_1=$	120.67564	98.068477
$\frac{\mathfrak{p}v}{4\omega_1}\frac{\theta_3''(0)}{\theta_3(0)}=$	26.73272	21.064838
Diff.$=I'=$	93.94292	77.003639
(other equat.)$=$	93.942921	77.003659

$$
\begin{array}{rll}
s = & -\ 0.33164947 & -\ 0.33084473 \\
-\frac{s}{q} = & 109.86656 & 74.300356 \\
1 + 2q^4 \cos 4\pi w = & 1.0000042 & 1.0000087 \\
b = & 109.86702 & 74.301000 \\
\sqrt{\frac{b-1}{b+1}} = & 0.99093909 & 0.98663068 \\
4q^2\sqrt{b^2-1}\,[1 - q^2(2b-1)] = & 0.00399641 & 0.00587502 \\
\text{Sum} = & 0.99493550 & 0.99250570 \\
I'' = & -\ 35.815107 & -\ 29.231710 \\
\frac{I}{2} = I' + I'' = & 58.127814 & 47.771939
\end{array}
$$

$$
\begin{aligned}
I_1 - I_2 &= 2(58.127814 - 47.771939) \\
&= 2(10.355875) \\
M &= 4\pi n_1 n_2 A a . 2(I_1 - I_2) \\
&= 4\pi n_1 n_2 (6213.52)
\end{aligned}
$$

In the calculation of I we have used the second value found for I' with $c = 110$ and the mean of the two values for I' with $c = 90$.

EXAMPLE 44. SHORT SECONDARY ON THE OUTSIDE OF A LONG PRIMARY

Length of primary $= 200$ cm
" " secondary $= 5$
Radius of primary $= 4 = a$
" " secondary $= 5 = A$

$$n_1 = 10 \qquad n_2 = 40$$

In Ròiti's formula:

$$
\begin{aligned}
\rho_1 &= 97.62811 \\
\rho_2 &= 102.62188 \\
\rho_2 - \rho_1 = \frac{xl}{\rho_1 + \rho_2} &= 4.9937586 \\
\frac{A^2 a^2}{8}\left(\frac{1}{\rho_1^3} - \frac{1}{\rho_2^3}\right) &= 0.0000075 \\
\text{Sum} &= 4.9937661 \\
4\pi^2 a^2 n_1 n_2 &= 25600\pi^2 \\
\therefore M &= 0.0012617342 \text{ henry.}
\end{aligned}
$$

In Searle and Airey's formula:

$$d^2 = 10025 \qquad N_1N_2 = 400000$$

$$\frac{l}{A} = 20 \qquad \frac{x}{a} = \frac{5}{8}$$

$$X_2 = -1597 \qquad L_2 = 1.4375$$

$$L_4 = -0.796$$

		1.00000000
1st term	=	0.00000071
2d "	=	-1.3×10^{-10}
Sum	=	1.00000071
$\therefore M$	=	0.0012617342 henry

in exact agreement with the above. Both formulas are very rapidly convergent, and give as nearly the same value for M as can be calculated with seven-place logarithms.

EXAMPLE 45. COILS OF EQUAL RADII NEAR TOGETHER, BY FORMULA (48)

Lengths of the coils 4 cm and 6 cm $= b_1$ and b_2
Radius " " " 20 cm $= a$
Distance between centers 10 cm $= d$
The calculation of the quantities in the parentheses is as follows:

	First	Second	Third	Fourth
1st term =	0.088055	0.01562	0.0047	.0022
2d term =	−0.012694	0.00126	−0.0002	
3d term =	0.001232	0.00130		
4th term =	−0.000104			
Sum =	0.076489	0.01818	0.0045	.0022

The expression for ΔM then gives:

First term	=0.0233240
Second "	=0.0011047
Third "	=0.0000973
Fourth "	=0.0000113
Sum	=0.0245373

The value of M_0 calculated for two circles of radius 20 cm and at a distance apart = 10 cm was found by (4) and checked by (1) to be

$$\frac{M_0}{N_1N_2} = 4\pi a\ (0.8853877)$$

$$\frac{\Delta M_0}{N_1N_2} = 4\pi a\ (0.0245373)$$

$$\therefore \quad \frac{M}{N_1N_2} = 4\pi a\ (0.9099250)$$

This was checked by means of (51) with the result, (assuming one turn per cm of the length of the coils)

$$\begin{aligned} L_{ABC} &= 4\pi a\ (430.339736) \\ L_C &= 4\pi a\ \ (74.324564) \\ \hline \text{Sum} &= 4\pi a\ (504.664300) \\ \\ L_{AC} &= 4\pi a\ (194.210135) \\ L_{BC} &= 4\pi a\ (266.777705) \\ \hline \text{Sum} &= 4\pi a\ (460.987840) \\ \frac{1}{2}\ \text{Diff.} &= 4\pi a\ \ (21.838230) \end{aligned}$$

Dividing by 24, the product of the number of turns assumed in calculating the self-inductances,

$$\frac{M}{N_1N_2} = 4\pi a\ (0.9099264)$$

which agrees with the value by (48) to about one and a half in a million.

For these coils, therefore, (48) is adequate to give a high degree of precision. If the distance between the same coils were, however, smaller, or if the lengths of the coils were greater the accuracy would not be so great, and it might be necessary to use (51). The latter, should, however, not be used when (48) converges well, since to get the same accuracy the calculation of the four self-inductances must be carried out to a greater number of decimal places than appear in the value of M. For the rapid convergence of (48) the ratios $\frac{b_1}{d}$, $\frac{b_2}{d}$ and $\frac{d}{a}$ should all be small.

For the more unfavorable case $b_1 = 6$, $b_2 = 10$, $d = 10$, $a = 20$ the value of ΔM comes out too small by three parts in ten thousand.

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EXAMPLE 46. ROSA-WEINSTEIN FORMULA (50). FOR COILS FARTHER APART

As a rather unfavorable case we may take

$$b_1 = 10 \qquad b_2 = 20 \qquad d = 50 \qquad a = 25$$

$$k = \quad \sin\gamma = \frac{50}{\sqrt{5000}} = \frac{1}{\sqrt{2}} = \cos\gamma$$

$$\frac{\cos^2\gamma}{24d^2} = \frac{\sin^2\gamma}{24d^2} = \frac{1}{120000}$$

$$\begin{array}{rl}
\alpha_1 = & 500 \\
-\alpha_3 = - & 4.55 \\
-3\alpha_3\cos^2\gamma = - & 6.825 \\
8\alpha_3\cos^4\gamma = + & 9.10 \\ \hline
\text{Sum} = & 497.725 \\
P = & .0041477
\end{array}
\qquad
\begin{array}{rl}
\alpha_1 = & 500 \\
2\alpha_3 = & 9.10 \\
+3\alpha_3\cos^2\gamma = + & 6.825 \\
8\alpha_3\cos^4\gamma = & 9.10 \\ \hline
\text{Sum} = & 525.025 \\
Q = & .0043752
\end{array}$$

$$\begin{array}{rl}
F = & 1.854075 \\
E = & 1.350644 \\ \hline
F - E = & 0.503431
\end{array}$$

$$(F-E)P + EQ = 0.0079974$$

$$\therefore \Delta M = 1.7766$$

$$\text{Terms in the 6th and 8th differentials} = 0.0016$$

$$\text{Sum} = 1.7782$$

From formula (19) which applies to the two circles at the centers of these coils

$$M_0 = 1.418599 \times 25 \quad = 35.4650$$

$$\therefore \frac{M}{N_1N_2} = 35.4650 + 1.7782 = 37.2432$$

If we calculate the mutual inductance by formula (51) we find

$$\begin{array}{rl}
L_{ABC} = & 3792.2261 \\
L_C = & 1667.7268 \\ \hline
 & 5459.9529
\end{array}
\qquad
\begin{array}{rl}
L_{AC} = & 2350.4870 \\
L_{BC} = & 3062.0405 \\ \hline
 & 5412.5275
\end{array}$$

$$\frac{1}{2}\text{ Diff.} = \quad 23.7127$$

$$\text{Dividing by } 200 = \quad 0.1185635 \quad = \frac{M}{4\pi a N_1 N_2}$$

$$\therefore \frac{M}{N_1N_2} = \quad 37.2478$$

which is more than one in ten thousand greater than the value by (50). If the coils had been shorter and their diameter had been greater than the distance between their medial planes, the quantities P and Q in (50) would have been more convergent and the value of ΔM would have been more nearly correct. The accuracy here obtained would, however, suffice in many cases.

This formula when applied to the coils in the preceding problem gives a very accurate result viz, $\frac{M}{N_1N_2}=4\pi a(0.909932)$, or about six in a million too large. (The terms in the sixth and eight order differentials as calculated by (48) are taken into account in this result.)

The mutual inductance of the coils in this example could also be calculated with a good degree of accuracy by Gray's formula.

EXAMPLE 47. METHOD OF OBTAINING THE DIMENSIONS OF THE EQUIVALENT CURRENT SHEETS

Suppose it is desired to obtain the mutual inductance of two solenoids, whose measured dimensions are as follows:

Coil I is wound with 100 turns of insulated wire of 0.15 cm covered diameter, the successive turns being in contact. The measured external diameter of the coil is 50.4 cm.

Coil II is wound with 50 turns of bare wire, 0.1 cm in diameter, in a thread of 2 mm pitch. The diameter measured over the wire is 10.25 cm.

Then the mean radius of coil I, to the center of the wire, is equal to $\frac{1}{2}(50.4-0.15)$ or 25.125 cm. The length of the equivalent current sheet will be the distance between the center of the first and the one hundred and first wire, or one hundred times the covered diameter of the wire; that is, 15 cm. Since the turns are in contact, the equivalent length may, in this case, also be found by measuring the over-all length of the winding, including the insulation. Both these methods are equivalent to taking one hundred times the pitch of the winding, which, in this case, is equal to the covered diameter of the wire.

For coil II, the equivalent radius is $\frac{1}{2}(10.25-0.1)=5.075$ cm. The equivalent length is fifty times $0.2=10$ cm.

The dimensions found in this way for coils I and II are to be used in the appropriate current sheet formula. (See p. 73.)

4. THE MUTUAL INDUCTANCE OF A CIRCLE AND A COAXIAL SINGLE-LAYER COIL

LORENZ'S FORMULA

The problem of finding the mutual inductance of a circle and a coaxial single layer winding was first solved by Lorenz.[53] Assuming that the current was uniformly distributed over the surface of the cylinder, forming a current sheet, he integrated over the length of the cylinder the expression for the mutual inductance of a circular element and the given circle. This expression is an elliptic integral. Lorenz reduced the integrated form to a series and gave the following formula for the mutual inductance of the disk and solenoid of what is now called the Lorenz apparatus. He called it, however, the constant of the apparatus instead of mutual inductance, and denoted it by C. It is of course the whole number of lines of magnetic force passing through the disk due to unit current in the surrounding solenoid.

$$M=\frac{\pi q r^2}{d}\Big[Q(\alpha_1)+Q(\alpha_2)\Big]$$

$$Q(\alpha)=2\pi q\sqrt{\frac{\alpha-1}{\alpha}}\Big[1+\frac{3}{8}\frac{q^2}{\alpha^2}+\frac{5}{16}\frac{q^4}{\alpha^4}\left(\frac{7}{4}-\alpha\right) + \frac{35}{128}\frac{q^6}{\alpha^6}\left(\frac{33}{8}-\frac{9}{2}\alpha+\alpha\right)+\cdots \qquad [53]$$

ρ = radius of the disk, Fig. 25.

r = radius of the solenoid.

$2x$ = length of winding of solenoid.

$q=\rho/r$ = ratio of the two radii.

$d=\frac{2x}{n}$ = distance between centers of successive turns of wire.

$\alpha=\frac{x^2+r^2}{r^2}$

---

[53] Wied. Annalen, **25**, p. 1; 1885. Oeuvres Scientifiques, **2**, p. 162.

If the disk be not exactly in the mean plane of the solenoid, and x_1 be the distance from the plane of the disk to one end of the solenoid and x_2 to the other,

$$\alpha_1 = \frac{x_1^2 + r^2}{r^2} \qquad \alpha_2 = \frac{x_2^2 + r^2}{r^2}$$

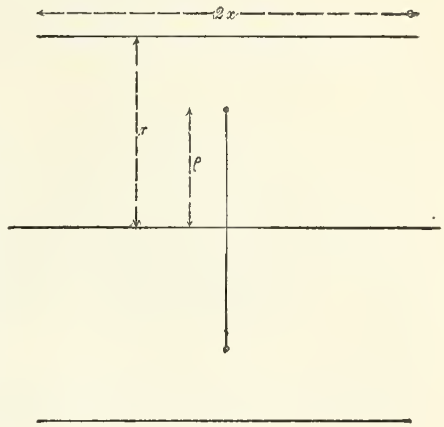

Fig. 25

Then $Q(\alpha_1)$ is found by substituting the values of α_1 in equation (53) above, and $Q(\alpha_2)$ by using the value of α_2 for α in the same equation. The sum of these two quantities multiplied by $\frac{\pi q r^2}{d}$ gives the constant of the instrument; that is, the mutual inductance sought.

As Lorenz gave the expression for the general term of (53), his equation can be extended. The following is the general term:

$$Q(\alpha) = 2\pi \sum_{m=0}^{m=\infty} q^{2m+1} \frac{1.3 \cdots 2m-1.}{2.4 \cdots \cdot 2m} \cdot \frac{1}{1.2 \cdots (m+1)} \cdot \frac{a^m}{da^m}\left(\frac{\alpha - 1}{\alpha}\right)^{m+\frac{1}{2}}$$

JONES'S FORMULAS

Two solutions of the above problem were given by Jones,[54] both in terms of elliptic integrals. The current was considered to flow not in a current sheet, but along a spiral winding or helix. The first solution was in the form of a series, convergent only when O_1A, Fig. 26, is less than the difference in the radii of inner and outer coils; that is, when O_1A is less than $A - a$. As this is a serious limitation, and the formula is a laborious one to use, it is not here given. The second solution is exact and general, and is in terms of elliptic integrals of all three kinds. The second formula is as follows:

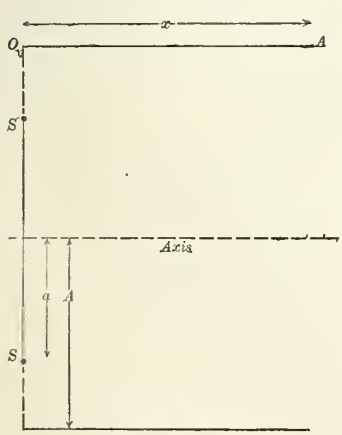

Fig. 26

$$M_\theta = \Theta(A+a)ck\left\{\frac{F-E}{k^2} + \frac{c'^2}{c^2}(F - \Pi)\right\} \qquad [54]$$

[54] J. V. Jones, Proc. Roy. Soc., 63, p. 198; 1898. Also, Trans. Roy. Soc., 182, A; 1891. Jones's first formula was given in Phil. Mag., 27, p. 61; 1889.

M_θ = mutual inductance of helix O_1A, Fig. 26, with respect to the disk S in the plane of one end.

$\Theta = 2\pi n$, $1/n$ = pitch of winding, Θ = whole angle of winding.

F, E, and Π are the complete elliptic integrals to modulus k, where

$$k^2 = \frac{4Aa}{(A+a)^2 + x^2} = \sin^2\gamma,\ c^2 = \frac{4Aa}{(A+a)^2},\ c'^2 = 1 - c^2.$$

Π, the complete elliptic integral of the third kind, can be expressed in terms of incomplete integrals of the first and second kinds, and the value of M_θ can then be calculated by the help of Legendre's tables. (See example 50.) The calculation is, however, extremely tedious, especially when the value is to be determined with high precision.

Campbell has given Jones's formula (54) a slightly different form,[55] somewhat more convenient in calculation, as follows:

$$M = 2\pi n_1 n_2 (A+a)\left\{\frac{c}{k}(F-E) + \frac{A-a}{b}\psi\right\} \qquad [55]$$

where n_1 is the same as n above, the number of turns per cm on the solenoid, n_2 is the number of turns in the secondary coil (in the above case it was taken as one), A is the greater and a the less of the two radii (in the above case A was the radius of the solenoid and a of the circle within), and

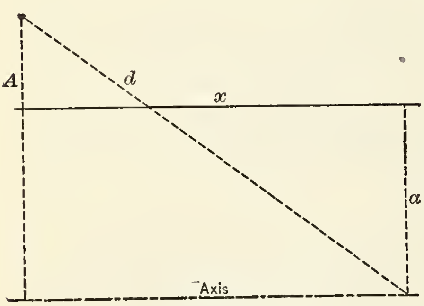

Fig. 27

$$\psi = F(k)E(k',\beta) - [F(k) - E(k)]F(k',\beta) - \frac{\pi}{2}$$

where $F(k)$ and $E(k)$ are the complete elliptic integrals to modulus k, and $F(k',\beta)$ and $E(k',\beta)$ are the incomplete elliptic integrals to modulus k' and amplitude β; $k' = \cos\gamma$, $\sin\beta = c'/k'$; k, c, and c' are given above. If a second-

[55] A. Campbell, Proc. Roy. Soc, A, 79, p. 428; 1907. There is a misprint in the formula as given in Campbell's paper. It was, however, used correctly in the numerical calculations given in the paper.

ary circle or coil has a radius greater than that of the solenoid, the same formula can be used if A is taken for the radius of the larger secondary and a is the radius of the solenoid (Fig. 27).

ROSA'S FORMULA[56]

The following formula gives the mutual inductance of a single layer coil of length x and a coaxial circle of radius a in the plane of one end of the coil, as shown in Fig. 26. It is the same quantity represented by M of equations (53) and (55) and M_θ of (54).

$$M_{OA}=\frac{2\pi^2a^2N}{d}\left[1+\frac{3}{8}\frac{a^2A^2}{d^4}+\frac{5}{64}\frac{a^4A^4}{d^8}X_2+\frac{35}{512}\frac{a^6A^6}{d^{12}}X_4+\frac{63}{1024}\frac{a^8A^8}{d^{16}}X_6\right.$$
$$\left.+\frac{231}{4096}\frac{a^{10}A^{10}}{d^{20}}X_8+\frac{429}{8192}\frac{a^{12}A^{12}}{d^{24}}X_{10}+\cdots\right] \quad [56]$$

$$X_2=3-4\frac{x^2}{A^2}$$

$$X_4=\frac{5}{2}-10\frac{x^2}{A^2}+4\frac{x^4}{A^4}$$

$$X_6=\frac{35}{16}-\frac{35}{2}\frac{x^2}{A^2}+21\frac{x^4}{A^4}-4\frac{x^6}{A^6}$$

$$X_8=\frac{63}{32}-\frac{105}{4}\frac{x^2}{A^2}+63\frac{x^4}{A^4}-36\frac{x^6}{A^6}+4\frac{x^8}{A^8}$$

$$X_{10}=\frac{231}{128}-\frac{1155}{32}\frac{x^2}{A^2}+\frac{1155}{8}\frac{x^4}{A^4}-165\frac{x^6}{A^6}+55\frac{x^8}{A^8}-4\frac{x^{10}}{A^{10}}$$

(For general coefficient, see p. 63.)
a = radius of disk or circle S, Fig. 26.
A = radius of the solenoid.
x = length O_1A of one end of the solenoid.
$d=\sqrt{x^2+A^2}$ = half the diagonal of the solenoid.
N is the whole number of turns of wire in the length x.

This formula is very easy to use in numerical calculation, notwithstanding it looks somewhat elaborate. The logarithm of $\frac{a^2A^2}{d^4}$, multiplied by 2, 3, 4, etc., gives the logarithm of the corresponding factor in each of the other terms. Similarly, the various terms X_2, X_4, etc., contain only powers of $\frac{x^2}{A^2}$ besides the numerical

[56] This Bulletin, 3, p. 209; 1907.

coefficients. It is hence a far simpler matter to compute M with high precision by this formula than by Jones's formula, the latter containing as it does elliptic integrals of all three kinds and involving the tedius interpolations for incomplete elliptic integrals.

If the secondary circle has a larger radius than the solenoid, A will be the radius of the circle and a the radius of solenoid. In every case A is the greater and a the less of the two radii, and d is $\sqrt{A^2+x^2}$.

Equation (56) may be written

$$M=\frac{2\pi^2 a^2 n_1 x}{d}S$$

where n_1 is the number of turns of wire per cm, x is the length of the coil, Fig. 26, and S is the value of the quantity in brackets in (56), which is always somewhat greater than unity. This may also be put as follows:

$$M=a^2 n_1\left(\frac{2\pi^2 x}{d}\right)S=a^2 n_1 RS$$

or, [57]

$$M=a^2 n_1 K$$

The quantity R depends on x/d; that is, only upon the shape of the solenoid. S depends upon x/A, a/A, and A/d; that is, upon the relative sizes of the inner circle and the solenoid and the shape of the solenoid. If we have the value of RS, or K of equation (57) for a given solenoid and circle, we can get M by multiplying by $a^2 n_1$, and for any other system of similar shape but different size by multiplying the same value of K by $a^2 n_1$. The values of the constant K for various values of a/A and x/A are given in Table III, page 193.

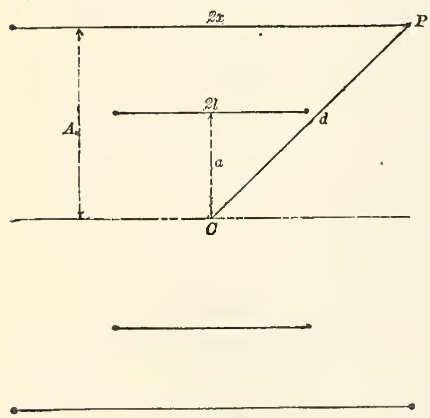

Fig. 28

If the disk or circle be in the center of a solenoid of length $2x$ (Fig. 28), the value of M is of course double that given by using x. If it be not quite in the center, the value of M must be calculated for each end separately.

For illustrations and tests of the above formulas, see examples 48 to 51, pages 103–110.

EXAMPLES ILLUSTRATING THE FORMULAS FOR THE MUTUAL INDUCTANCE OF A CIRCLE AND A COAXIAL SOLENOID

EXAMPLE 48. ROSA'S FORMULA (56) COMPARED WITH JONES'S FORMULA (54)

Professor Jones gave the calculations by formula (54) of the constant of the Lorenz apparatus made for McGill University, obtaining the values given below, the second value being that obtained after the plate had been reground and again measured.

A calculation[57] of the same two cases by formula (56) gives very closely agreeing results.

	1st value	2nd value, disk slightly smaller
By formula (54) $M=$	18,056.36	18,042.52
" " (56) $M=$	18,056.46	18,042.74
Difference	$-.10$	$-.22$

These differences, amounting to five parts in a million in the first case and twelve parts in a million in the second case, are wholly negligible in the most refined experimental work.

EXAMPLE 49. FORMULA (56) COMPARED WITH JONES'S FIRST FORMULA

Take as a second example the case given by Jones[58] to illustrate his first formula.

$$A = 10 \text{ inches} \qquad a = 5 \text{ inches} \qquad x = 2 \text{ inches}$$

$$d^2 = 104 \qquad \frac{a^2A^2}{d^4} = \frac{2500}{10816} \qquad \log \frac{a^2A^2}{d^4} = \bar{1}.3638733$$

	1st term	= 1.0000000
	2 "	= .0866771
$X_2 = 2.8400$	3 "	= .0118537
$X_4 = 2.1064$	4 "	= .0017781
$X_6 = 1.5208$	5 "	= .0002670
$X_8 = 1.0173$	6 "	= .0000379
$X_{10} = 0.5815$	7 "	= .0000046
$\frac{2\pi^2 a^2}{d} = 48.38972$	Sum	= 1.1006184

[57] This Bulletin, 3, p. 218; 1907.

[58] Phil. Mag., 27, p. 61; 1889. In this example, P_0 should be 0.654870 instead of 0.54870, as printed in Jones's article.

$\therefore M = 53.25861\ N$, N being the number of turns of wire on the coil.

Jones gives $M = 53.25879\ N$.

The difference between these values is three parts in a million.

EXAMPLE 50. CALCULATION OF CONSTANT OF AYRTON-JONES CURRENT BALANCE BY FORMULAS (54) AND (56)

As a further test of the formulas let us calculate the constant of an electro-dynamometer or current balance of the Ayrton-Jones type,[59] of which AB, Fig. 29, is the upper fixed coil and ED is the moving coil, the circle S at the upper end lying in the plane through the middle of AB and the circle R at the lower end of ED lying in the middle plane of the lower fixed coil BC.

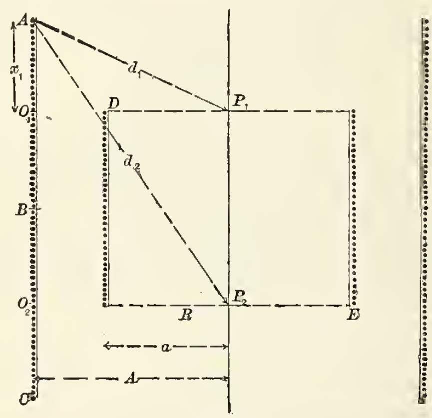

Fig. 29

Assume the dimensions as follows:

$A =$ 16 cm = radius of fixed coil, Fig. 29.
$a =$ 10 cm = radius of moving coil.
$x_1 =$ 8 cm = half length of AB = O_1A
$x_2 =$ 24 cm = 1.5 times AB = O_2A
$n_1 =$ 10 = number of turns per cm
$N_1 =$ 80 = number of turns in distance $O_1A = x_1$, Fig. 29.
$N_2 =$ 240 = number of turns in distance $O_2A = x_2$

[59] This Bulletin, 3, p. 226; 1907.

$$d_1 = \sqrt{A^2 + x^2_1} = 8\sqrt{5} = \text{diagonal AP}_1\text{, Fig. 29.}$$

$$d_2 = \sqrt{A^2 + x^2_2} = 8\sqrt{13} = \text{diagonal AP}_2$$

We have to determine two mutual inductances, namely, M_S between the coil O_1A of 80 turns on the circle S, and M_R between the coil O_2A of 240 turns on the circle R. In each case the circle is in the plane passing through the lower end of the coil.

Formula (56) will be used, taking N_1, x_1, and d_1 in the first case and N_2, x_2, and d_2 in the second case.

	For M_S	For M_R
A	16 cm	16 cm
a	10	10
x	8	24
A^2	256	256
x^2	64	576
$N=nx$	80	240
d^2	320	832
$\log d^2$	2.5051500	2.9201233
$\log \frac{a^2A}{d^4}$	$\bar{1}.3979400$	$\bar{2}.5679934$
$\log \frac{x^2}{A^2}$	$\bar{1}.3979400$	0.1760913
X_2	+2.000	− 6.00
X_4	+0.250	+ 0.25
X_6	−0.9375	+23.5
X_8	−1.203	−45.7
X_{10}	−0.562	−49.0
1st term	1.0000000	1.0000000
2d "	+ .0937500	+ .0138683
3d "	+ .0097656	− .0006411
4th "	+ .0002670	+ .0000009
5th "	− .0002253	+ .0000027
6th "	− .0000662	− .0000002
7th "	− .0000072	.0000000
Sum $=S$	1.1034839	1.0132306

$\log S_1$	=	0.0427660	$\log S_2$	=	0.0057083
" $2\pi^2$	=	1.2953298	" $2\pi^2$	=	1.2953298
" $a^2(=100)$	=	2.0000000	" $a^2(=100)$	=	2.0000000
" $N_1(=80)$	=	1.9030900	" $N_2(=240)$	=	2.3802112
		5.2411858			5.6812493
" d_1	=	1.2525750	" d_2	=	1.4600616
$\log M_s$	=	3.9886108	$\log M_R$	=	4.2211877
$\therefore M_s$	=	9741.16	M_R	=	16641.32

THE SAME EXAMPLE BY JONES'S FORMULA

We will now calculate M_S and M_R by Jones's second formula given above, using also the following equation to find $F-\Pi$:

$$\frac{k'^2 \sin\beta\cos\beta(F-\Pi)}{c} = F(k)E(k',\beta) + E(k)F(k',\beta) - F(k)F(k',\beta) - \frac{\pi}{2}$$

	For M_S	For M_R
A	16 cm	16 cm
a	10	10
x	8	24
$\Theta = 2\pi N$	$160\ \pi$	$480\ \pi$
$c = \dfrac{2\sqrt{Aa}}{A+a}$	0.9730085	0.9730085
$c' = \sqrt{1-c^2}$	0.2307692	0.2307692
$k = \dfrac{2\sqrt{Aa}}{\sqrt{(A+a)^2+x^2}}$	0.9299812	0.7149701
$k' = \sqrt{1-k^2}$	0.3676073	0.6991550
$\log \sin\beta\left(\sin\beta = \dfrac{c'}{k'}\right)$	9.7977938	9.5186043
$F(k)$	2.4373371	1.8636661
$E(k)$	1.1323456	1.3449927
$\dfrac{F-E}{k^2}$	1.5088957	1.0146546
$F(k', \beta)$	0.6852557	0.3394833
$E(k', \beta)$	0.6721988	0.3333201
$\dfrac{k'^2 \sin\beta\cos\beta(F-\Pi)}{c}$	−0.8266738	−1.1256799
$\dfrac{c'^2}{c^2}(F-\Pi)$	−0.6851799	−0.4045298

$\log\left\{\frac{F-E}{k^2}+\frac{c'^2}{c^2}(F-\Pi)\right\}$	$\bar{1}.9157773$	$\bar{1}.7854187$
$\log\left(\Theta(A+a)ck\right)$	4.0728340	4.4357689
$\log M$	3.9886113	4.2211876
	$M_S=9741.17$ cm	$M_R=16641.32$

M_S differs from the value obtained by formula (16) by one part in a million, M_R is identical.

M_S is the mutual inductance of the winding O_1A on S. The inductance M_1 of the whole coil AB on S is twice as much, that is

$$M_1=19482.34$$

The inductance of AB on R is M_R above, minus the inductance of O_2B on R which is the same as that of O_1A on S, that is, M_S. Therefore,

$$M_2=16641.32-9741.17=6900.15$$

Hence $M_1-M_2=12582.19$ cm.

The force of attraction of the one winding AB in dynes is

$$\frac{1}{2}f=i_1i_2n_2(M_1-M_2).$$

The force due to the second winding BC is equal to this. Suppose $i_1=i_2=1$ ampere $=0.1$ c.g.s. unit of current and $n_2=10$ turns per cm. Then

$$i_1i_2n_2=0.10$$

$$\begin{aligned}\therefore f&=0.20\times 12582.19 \text{ dynes}\\ &=2516.438 \text{ dynes}\\ 2f&=5032.876 \text{ dynes}=\text{change of force on reversal of current}\\ &=5.1356 \text{ gms where } g=980.\end{aligned}$$

If there are two sets of coils, one on each side of the balance, as in the ampere balance built for the National Physical Laboratory, the force would be doubled again.

In calculating the mutual inductance of the disk and surrounding solenoid in the Lorenz apparatus the series (56) will be more convergent when the winding is long, and of course more convergent when the disk is not of too great diameter.

EXAMPLE 51. MUTUAL INDUCTANCE OF CAMPBELL'S FORM OF STANDARD BY FORMULAS (55) AND (56)

A cylinder 20 cm in diameter has two coils of 50 turns each wound as shown in Fig. 30, each covering 5 cm (=AB) with an interval of 10 cm between (=AA′). A secondary coil of 1000 turns of finer wire is wound in a channel S, with a mean radius of 14.5 cm. The magnetic field near S, due to the double solenoid, is very weak, and is zero at some point; at this place M will be a maximum, and variations in M due to small changes in A will be very small. To calculate M for the solenoid AB and the coil S, we take two cases, as in the preceding example. First, M for S and a winding O_2B of 100 turns; second, M for S and O_2A of 50 turns. The difference will be M for S and the actual winding AB. Or, supposing

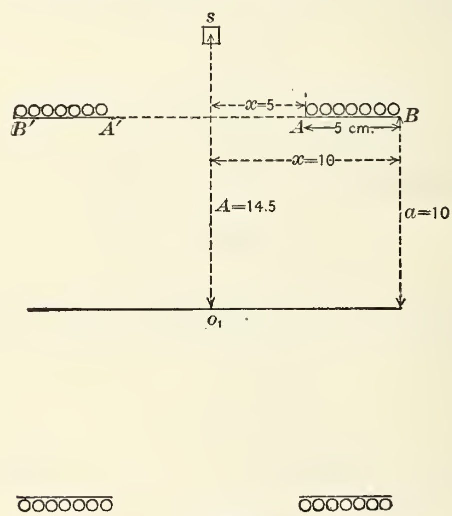

Fig. 30

AB to have 100 turns, M will be the same as for AB of 50 and A′ B′ of 50. Using formula (55) we have the following values:

	For M_1	For M_2
$a=$	10	$=$ 10
$A=$	14.5	$=$ 14.5
$x=b=$	10	$=$ 5.0
$\log k=$	$\bar{1}.9590874$	$= \bar{1}.98366715$
$\gamma=$	$65°31'7''.32$	$= 74°23'38''.88$
$k'=$	$\sqrt{0.1717243}$	$= \sqrt{0.0723711}$
$\beta=$	$26°18'36''.85$	$= 43°3'33''.06$
$\gamma'=$	$24°28'52''.68$	$= 15°36'21''.12$
$F=$	2.3267801	$=$ 2.7312000
$E=$	1.1590043	$=$ 1.0812388
$\frac{c}{k}(F-E)=$	1.2613045	$=$ 1.6839704
$F(k',\beta)=$	0.4618972	$=$ 0.7561693
$E(k',\beta)=$	0.4565314	$=$ 0.7469284
$\psi=$	-1.0479404	$= -0.7784352$
$\frac{A-a}{b}\psi=$	-0.4715732	$= -0.7005918$
$\frac{c}{k}(F-E)+\frac{A-a}{b}\psi=$	0.7897313	$=$ 0.9833786
$n_1n_2=$	200,000	$=$ 100,000
$M_1=$	24,313,940 cm	$M_2=$ 15,137,940 cm
$=$	24.31394 millihenrys	$=$ 15.13794 millihenrys

$M= M_1-M_2=9.1760$ millihenrys.

Campbell gives[60] the value of M as 9.1762 millihenrys, but for want of any particulars of his calculation we do not know wherein the difference lies.

We have worked this problem out also by formula (56) with the following results:

$$\begin{aligned} M_1 &= 24.31387 \text{ millihenrys} \\ M_2 &= \underline{15.13857} \quad \text{"} \quad \text{"} \\ M &= 9.17530 \quad \text{"} \quad \text{"} \end{aligned}$$

The value of M_1 agrees with that found by (55) to about two parts in a million. M_2 is, however, a little larger, making M smaller. This is due to the fact that formula (56) is not as convergent for

[60] A. Campbell, Proc. Roy. Soc., **79**, p. 428; 1907.

$x=5$ in this problem as for $x=10$, and hence the terms neglected after the seventh are appreciable. Hence, for so short a coil as this, formula (54) or (55) will give a more accurate result than (56).

5. THE SELF-INDUCTANCE OF A CIRCULAR RING OF CIRCULAR SECTION

KIRCHHOFF'S FORMULA

The formula for the self-inductance of a *circle* was first given by Kirchhoff[61] in the following form:

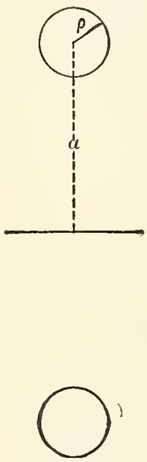

Fig. 31

$$L=2l\left\{\log\frac{l}{\rho}-1.508\right\} \qquad [58]$$

where l is the circumference of the circular conductor and ρ is the radius of its cross section. This is equivalent to the following:

$$L=4\pi a\left\{\log\frac{8a}{\rho}-1.75\right\} \qquad [59]$$

a being the radius of the circle, Fig. 31. These formulas are approximate, being more nearly correct as the ratio ρ/a is smaller

MAXWELL'S FORMULA

A more accurate expression, obtained by means of Maxwell's principle of the geometrical mean distance, is the following:

$$L=4\pi a\left\{\left(1+\frac{3}{16}\frac{R^2}{a^2}\right)\log\frac{8a}{R}-\left(2+\frac{R^2}{16a^2}\right)\right\} \qquad [60]$$

Substituting in this equation the value of the geometrical mean distance for a circular area, $R=\rho e^{-\frac{1}{4}}=.7788\rho$, we obtain[62]

$$L=4\pi a\left\{\left(1+0.1137\frac{\rho^2}{a^2}\right)\log\frac{8a}{\rho}-.0095\frac{\rho^2}{a^2}-1.75\right\} \qquad [61]$$

This is a very accurate formula for circles in which the radius of section ρ is very small in comparison with the radius a of the circle. The geometrical mean distance R has, however, been computed on the supposition of a linear conductor, and can only

[61] Pogg. Annalen, **121**, p. 551; 1864.

[62] Wied. Annalen, **53**, p. 935; 1894.

be applied to circles of relatively small value of ρ/a, and the square of the geometrical mean distance is used for the arithmetical mean square distance in the second order terms. We must therefore expect an appreciable error in formula (61) when the ratio ρ/a is not very small. Formulas (58), (59), and (61) have been deduced on the supposition of a uniform distribution of the current over the cross section of the ring.

If the ring is a hollow, circular, thin tube, or if the current in the ring is alternating and of extremely high frequency, so that it can be regarded as flowing on the surface of the ring, the geometrical mean distance for the section would be the radius ρ, and we should have instead of (61) the following by substituting $R=\rho$,

$$L=4\pi a\left\{\left(1+\frac{3}{16}\frac{\rho^2}{a^2}\right)\log\frac{8a}{\rho}-\frac{\rho^2}{16a^2}-2\right\} \qquad [62]$$

In the case of solid rings carrying alternating currents of moderate frequency the value of L would be somewhere between the values given by (61) and (62).

RAYLEIGH AND NIVEN'S FORMULA

Rayleigh and Niven gave,[63] without proof, the following formula for a *circular coil* of n turns and of circular section,[64] which is more nearly exact than either of the preceding:

$$L=4\pi n^2 a\left\{\left(1+\frac{\rho^2}{8a^2}\right)\log\frac{8a}{\rho}+\frac{\rho^2}{24a^2}-1.75\right\} \qquad [63]$$

When $n=1$, this will be the self-inductance of a single circular ring.[65] This formula neglects higher powers of $\frac{\rho}{a}$ than the second,

[63] Rayleigh's Collected Papers, Vol. II, p. 15.

[64] Neglecting the correction for effect of insulation and shape of section of the separate wires.

[65] Max Wien, Wied. Annalen, 53, p. 928, 1894, derived by direct integration of Maxwell's formula (12) over the cross section of the ring, the formula

$$L=4\pi a\left\{\left(1+\frac{\rho^2}{8a^2}\right)\log\frac{8a}{\rho}-0.0083\frac{\rho^2}{a^2}-1.75\right\}$$

It was shown, however, by Terezawa, Tokyo Math. Phys. Soc., 5, p. 84, 1909, that this formula is in error, the correct result being identical with that of Rayleigh and Niven (63). This result was verified by Mr. Cohen at the Bureau of Standards in 1909, and quite recently independently by Mr. T. J. Bromwich of Cambridge, England. The error of Wien's expression is in practical cases of no importance.

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and its error therefore depends on the magnitude of the ratio of the radius of the cross section to the radius of the ring. Assuming, as is probably justified, that the coefficients of the terms in $\left(\frac{\rho}{a}\right)^4$, are of the same magnitude, or smaller, than those of the terms in $\left(\frac{\rho}{a}\right)^2$, the error will not be greater than $\frac{1}{100000}$ even for $\frac{\rho}{a}=0.1$, an exceptionally unfavorable case.

If used for a coil of more than one turn, the expression for L must be corrected for the space occupied by the insulation between the wires and for the shape of the section.[66]

SELF-INDUCTANCE OF A TUBE BENT INTO A CIRCLE

Suppose that the cross section of the ring is not solid, but is an annulus bounded by two concentric circles of radii ρ_1 and ρ_2, ρ_2 being the larger. Then assuming the current to be uniformly distributed over the cross section, we find[67] by means of Wien's method

$$
\begin{aligned}
L = 4\pi a\Bigg[&\left(1+\frac{\rho_1^2+\rho_2^2}{8a^2}\right)\log\frac{8a}{\rho_2} - 1.75 + \frac{2\rho_2^2+\rho_1^2}{32a^2} \\
&-\frac{\rho_1^2}{2(\rho_2^2-\rho_1^2)} + \frac{\rho_1^4}{(\rho_2^2-\rho_1^2)^2}\left(1+\frac{\rho_1^2}{8a^2}\right)\log\frac{\rho_2}{\rho_1} \\
&-\frac{\rho_2^4+\rho_1^2\rho_2^2+\rho_2^4}{48a^2(\rho_2^2-\rho_1^2)}\Bigg]
\end{aligned}
\qquad [64]
$$

In this formula terms of higher order than $\frac{\rho_2^2}{a^2}$ and $\frac{\rho_1^2}{a^2}$ have been neglected. Expanding (64) in terms of $\frac{\rho_2^2-\rho_1^2}{a^2}$ and letting ρ_1 approach ρ_2 we find for the case of a tube with infinitely thin walls, or of a tube carrying a current of infinitely high frequency,

$$
L = 4\pi a\left[\left(1+\frac{\rho^2}{4a^2}\right)\log\frac{8a}{\rho} - 2\right] \qquad [65]
$$

---

[66] See Rosa, this Bulletin, 3, p. 1; 1907.

[67] Grover, Phys. Rev., 30, p. 787; 1910.

a result which was also found by direct integration,[68] and which was subsequently communicated to us by Mr. T. J. Bromwich.

This corresponds to Maxwell's equation (62), but as might be expected gives a slightly greater value for the inductance.

If we expand (64) in terms of $\frac{\rho_1}{\rho_2}$ and let ρ_1 approach zero, we find for the limiting case of a ring with a solid cross section, the same formula (63) as was derived by directly performing the integration for this case.

An important case is that of a ring of solid cross section, where the current is not distributed uniformly over the cross section, but the current density is proportional to the distance from the axis of the ring. This would apply to the case of a ring revolving about a diameter in a uniform magnetic field. For this Wien (loc. cit.) derived the formula

$$L=4\pi a\left\{\left(1+\frac{3}{8}\frac{\rho^2}{a^2}\right)\log\frac{8a}{\rho}-.092\frac{\rho^2}{a^2}-1.75\right\} \qquad [66]$$

J. J. THOMSON'S FORMULA FOR RING OF ELLIPTICAL SECTION

If the circular ring has an elliptical section the approximate formula for its self-inductance (corresponding to (59) for a circular section) is[69]

$$L=4\pi a\left\{\log\frac{16a}{\alpha+\beta}-1.75\right\} \qquad [67]$$

where α and β are the semiaxes of the ellipse, and a is the mean radius of the circular ring.

The formulas of Minchin,[70] Hicks,[71] and Bláthy[72] we have elsewhere[73] shown to be incorrect, and hence they are not here given.

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[68] Russell also gives equation (65) but without the term in $\frac{\rho^2}{a^2}$ in Phil. Mag., **13**, p. 430; 1907.

[69] J. J. Thomson, Phil. Mag., **23**, p. 384; 1886.

[70] Phil. Mag., **37**, p. 300; 1894.

[71] Phil. Mag.. **38**, p. 456; 1894.

[72] London Electrician, **24**, p. 630; Apr. 25, 1890.

[73] This Bulletin, **4**, p. 149; 1907.

EXAMPLES ILLUSTRATING THE FORMULAS FOR THE SELF-INDUCTANCE OF CIRCULAR RINGS OF CIRCULAR SECTION

EXAMPLE 52. COMPARISON OF FOUR FORMULAS FOR THE SELF-INDUCTANCE OF CIRCLES

For a circle of radius $a=25$ cm and $\rho=0.05$ cm we obtain from the four formulas the following values of L:

By Kirchhoff's formula (59)	$L=654.40496\pi$ cm
By Maxwell's formula (61)	$L=654.40533\pi$ cm
By Rayleigh and Niven's (63)	$L=654.40548\pi$ cm
By Wien's second formula (66)	$L=654.40617\pi$ cm.

Thus for so small a value of $\frac{\rho}{a}$ as 1/500 any of these formulas is sufficiently accurate, the greatest difference being less than one in a million, except in the case of formula (66).

EXAMPLE 53. SECOND COMPARISON OF FOUR FORMULAS FOR CIRCLES

For a circle of radius $a=25$ cm, $\rho=0.5$ cm, $\frac{\rho}{a}$ being 1/50.

By Kirchhoff's formula (59)	$L=424.1464\pi$ cm
By Maxwell's formula (61)	$L=424.1734\pi$ cm
By Rayleigh and Niven's formula (63)	$L=424.1781\pi$ cm
By Wien's second formula (66)	$L=424.2326\pi$ cm.

EXAMPLE 54. THIRD COMPARISON OF FOUR FORMULAS FOR CIRCLES

For a circle of radius $a=10$ cm, $\rho=1.0$, $\frac{\rho}{a}=1/10$.

By Kirchhoff's formula (59)	$L=105.281\pi$ cm
By Maxwell's formula (61)	$L=105.476\pi$ cm
By Rayleigh and Niven's formula (63)	$L=105.517\pi$ cm
By Wien's second formula (66)	$L=105.902\pi$ cm.

It will be seen that for the smallest ring of radius 10 cm and diameter of section 2 cm Maxwell's formula gives a result 1 part in 2500 too small, while the simple approximate formula of Kirchhoff is in error by one in four hundred. For the larger rings the differences are much smaller.

Wien's second formula gives appreciably larger values than the others, as it should do.

EXAMPLE 55. COMPARISON OF FORMULAS (62) AND (65) FOR VERY THIN WALLED TUBES

(*a*) $a=25$ $\rho=0.05$ cm

By Maxwell's formula (62)	$L=629.40556\pi$ cm
By Formula (65)	$L=629.40579\pi$ cm
Solid ring (63)	$L=654.40548\pi$ cm

(*b*) $a=25$ $\rho=0.5$ cm

By Maxwell's formula (62)	$L=399.1889\pi$ cm
By Formula (65)	$L=399.2064\pi$ cm
Solid ring (63)	$L=424.1781\pi$ cm

(*c*) $a=10$ $\rho=1.0$ cm

By Maxwell's formula (62)	$L=95.585\pi$ cm
By Formula (65)	$L=95.719\pi$ cm
Solid ring (63)	$L=105.517\pi$ cm.

Maxwell's expression is nearly correct for the larger ring, but the error increases rapidly as the ratio $\frac{\rho}{a}$ is increased.

EXAMPLE 56. FORMULA (64) FOR A TUBULAR RING

$a=20$ $\rho_2=0.5$ cm = external radius of the cross section.

The calculation has been carried through for different thicknesses of the walls of the tube $(\rho_2-\rho_1)$ ranging from zero (infinitely thin-walled tube) to ρ_2 (solid cross section).

ρ_1	$\frac{\rho_1}{\rho_2}$		L cm
0	0	Solid ring	1010.032
0.125	¼		1003.210
0.25	½		987.528
0.375	¾		968.045
0.5	1	Infinitely thin walls	947.308

In formula (64), next to the first two terms, the fourth and fifth terms are the most important.

6. THE SELF-INDUCTANCE OF A SINGLE LAYER COIL OR SOLENOID

The following approximate formula for the self-inductance of a long solenoid is often given:

$$L = 4\pi^2 a^2 n_1^2 b \qquad [68]$$

where a is the mean radius, n_1 is the number of turns of wire per cm, and b is the length, supposed great in comparison with a. There is a considerable error in this formula, due to the end effect, but the variations in L due to changes in l are almost exactly proportional to the changes in l, and hence this formula may be used for calculating the corresponding variations in L.

RAYLEIGH AND NIVEN'S FORMULAS

The following formula[74] for the self-inductance of a single layer winding on a solenoid is very accurate when the length b is small compared with the radius a, Fig. 32:

$$L_s = 4\pi a n^2 \left\{ \log \frac{8a}{b} - \frac{1}{2} + \frac{b^2}{32a^2}\left(\log \frac{8a}{b} + \frac{1}{4} \right) \right\} \qquad [69]$$

n is the whole number of turns of wire on the coil, and the radius is measured to the center of the wire. The length b is the *mean over-all length including the insulation on the first and last wires* if the coil is wound closely with insulated wire. (See also p. 97.)

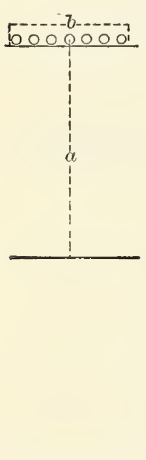

Fig. 32

The self-inductance L_s is, however, not the actual self-inductance of the coil, but the current sheet value; that is, it is the value of the self-inductance if the winding were of infinitely thin tape, so that the current would cover the entire length b. To get the actual self-inductance L for any given case one must correct L_s by formula (80) below. The same remark applies to all the formulas in this section for L_s. The approximate formula (68) is too rough to make it worth while to apply such a correction.

For a coil in which the axial dimension b is zero and the radial depth is c, the following current sheet formula of Rayleigh and Niven gives the self-inductance:

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[74] Proc. Roy. Soc., **32**, pp. 104–141; 1881. Rayleigh's Collected Papers, **2**, p. 15.

$$L_s = 4\pi a n^2\left\{\log\frac{8a}{c} - \frac{1}{2} + \frac{c^2}{96a^2}\left(\log\frac{8a}{c} + \frac{43}{12}\right)\right\} \qquad [70]$$

This is not an important case in practice.

Formulas (69) and (70) may be obtained from (88) by making first $c=0$ and then $b=0$.

COFFIN'S FORMULA

Coffin[75] has extended formula (69) so that it is very accurate for coils of length as great as the radius, and sufficiently accurate for most purposes for coils considerably longer than this.

$$L_s = 4\pi a n^2\left\{\log\frac{8a}{b} - \frac{1}{2} + \frac{b^2}{32a^2}\left(\log\frac{8a}{b} + \frac{1}{4}\right) - \frac{1}{1024}\frac{b^4}{a^4}\left(\log\frac{8a}{b} - \frac{2}{3}\right)\right.$$
$$\left. + \frac{10}{131072}\frac{b^6}{a^6}\left(\log\frac{8a}{b} - \frac{109}{120}\right) - \frac{35}{4194304}\frac{b^8}{a^8}\left(\log\frac{8a}{b} - \frac{431}{420}\right)\right\} \qquad [71]$$

LORENZ'S FORMULA

Lorenz first gave[76] an exact formula for the self-inductance of a

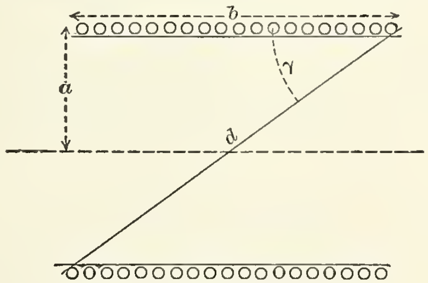

Fig 33

single layer solenoid. It is, like the others, a current sheet formula, and requires correction by (80) for a winding of wire, but applies to a solenoid of any length. Changing the notation slightly Lorenz's formula as originally given is as follows:

[75] This Bulletin, 2, p. 113; 1906.

[76] Wied. Annal., 7, p. 161; 1879. Oeuvres Scientifiques de L. Lorenz, Tome, 2, 1, p. 196.

$$L_s = \frac{32}{3}\frac{\pi n^2 a^3}{b^2}\left\{\frac{2k^2-1}{k^3}E + \frac{1-k^2}{k^3}F - 1\right\} \qquad [72]$$

where $k^2 = \dfrac{4a^2}{4a^2+b^2}$ and F and E are complete elliptic integrals of the first and second kind of modulus k, and a, b, and n are the radius (Fig. 33), length, and whole number of turns of wire, respectively. By simple substitutions the formula may be put into the following form, where d is the diagonal of the solenoid $= \sqrt{4a^2+b^2}$;

$$L_s = \frac{4\pi n^2}{3b^2}\left\{d(4a^2-b^2)E + db^2F - 8a^3\right\} \qquad [73]$$

Coffin derived[77] an expression for L in elliptic integrals which is equivalent to (73), and also obtained (73) from an expression[78] attributed to Kirchhoff.

Formula (73) may be written

$$L_s = an^2\left[\frac{8\pi}{3}\left\{\sqrt{1+\frac{b^2}{4a^2}}\left(\frac{4a^2}{b^2}-1\right)E + \sqrt{1+\frac{b^2}{4a^2}}F - \frac{4a^2}{b^2}\right\}\right] \qquad [74]$$

or $\qquad L_s = an^2Q$

where a is the radius of the solenoid, n is the whole number of turns on the coil, and Q is the function of $\dfrac{2a}{b}$ $(=\tan\gamma)$ contained in the square brackets. We have calculated Q for various values of $\tan\gamma$ from 0.2 to 4.0 and given them in Table IV, page 194. This table will be found useful in calculating L_s for solenoids when $\tan\gamma$ has one of the values given in the table, as all calculation of elliptic integrals is avoided. In problems where the length and diameter can be chosen at will, as in the designing of apparatus, this method of calculating L will be most frequently useful. The values of the constant Q given in the table have been computed with great care, so that they give very accurate values of L_s, for long as well as short solenoids.

In calculating the value of L_s by means of formula (69), (71), (73), or (74) and the following, one should use for the length b the *over-all*

[77] This Bulletin, 2, p. 123, equation (31); 1906.

[78] This Bulletin, 2, p. 127, equation (36). The notation is slightly different.

length including the insulation (A B, Fig. 34, and not *a b*) for a close winding of insulated wire, or *n times the pitch* for a uniform winding of bare or covered wire, which is, of course, the same as the length from center to center of $n+1$ turns. The radius a is the mean radius to the center of the wire. The same method of taking the breadth and depth b and c applies in the formulas of section 7. (See also remarks under example 47.)

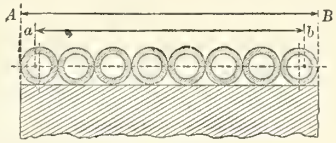

Fig. 34

NAGAOKA'S FORMULAS AND TABLES

In a recent paper[79] Nagaoka has derived formulas and prepared tables by which the self-inductance of a cylindrical current sheet of any dimensions whatever may be accurately and conveniently calculated. Starting from his absolute formula (45) for the mutual inductance of coaxial solenoids, he passes to the special case that the two solenoids coincide, and shows that the resulting expression for the self-inductance is equivalent to Lorenz's absolute formula (73), which he then expands in terms of q or q_1 functions.

He expresses the inductance of a coil of finite length by means of the expression (68) for an infinitely long coil, introducing a correction factor K, which is less than unity, to take account of the effect of the ends of the coil.

Thus

$$L = 4\pi^2 a^2 n_1^2 b K = 4\pi^2 a^2 \frac{n^2}{b} K \qquad [75]$$

where K is a function of half the angular aperture θ of the coil at the center. Nagaoka has prepared tables giving K with θ as argument and also as function of the $\frac{\text{diameter}}{\text{length}} = \frac{2a}{b}$. These tables are reproduced here as Tables XX and XXI, and enable K to be obtained by interpolation with all the accuracy that will usually be required. In case, however, it becomes necessary to obtain a more accurate value of K than can be obtained from these tables, or in such cases as fall outside the range of the tables, or in a portion where the function is changing so rapidly as to make interpolation difficult, the following formulas may be used to calculate K directly.

[79] Jour. Coll. Sci. Tokyo, 27, art. 6, pp. 18–33; 1909.

For short solenoids

$$K=\frac{1}{3\pi\sqrt{q_1}(1+\alpha_1)^2}\left[1-\frac{k'^2}{k^2}+\left\{\frac{k'^2}{k^2}\left(1+\frac{8\beta_1}{1+\alpha_1}\right)+\frac{8\gamma_1}{1-\delta_1}\right\}\frac{1}{2}\log_e\frac{1}{q_1}\right]-\frac{4}{3\pi}\frac{k}{k'}$$

where [76]

$$\alpha_1=q_1^2+q_1^6+q_1^{12}+\cdots$$

$$\beta_1=q_1^2+3q_1^6+6q_1^{12}+\cdots \qquad k^2=\frac{4a^2}{4a^2+b^2}$$

$$\gamma_1=q_1-4q_1^4+9q_1^9-\cdots \qquad k'^2=\frac{b^2}{4a^2+b^2}$$

$$\delta_1=2q_1-2q_1^4+2q_1^9-\cdots$$

$$q_1=\frac{l_1}{2}+2\left(\frac{l_1}{2}\right)^5+15\left(\frac{l_1}{2}\right)^9+\cdots$$

$$l_1=\frac{1-\sqrt{k}}{1+\sqrt{k}}=\frac{k'^2}{(1+k)(1+\sqrt{k})^2}$$

For relatively long coils

$$K=\frac{2}{3(1-\delta)^2}\left\{1+\frac{8\beta}{1+\alpha}+\frac{k'^2}{k^2}\cdot\frac{8\gamma}{1-\delta}\right\}-\frac{4}{3\pi}\frac{k}{k'} \qquad [77]$$

where k and k' have the same values as in (76) and

$$q=\frac{l}{2}+2\left(\frac{l}{2}\right)^5+15\left(\frac{l}{2}\right)^9+\cdots$$

$$l=\frac{1-\sqrt{k'}}{1+\sqrt{k'}}=\frac{k^2}{(1+k')(1+\sqrt{k'})^2}$$

and α, β, γ, δ are given by the same equations as α_1, β_1, γ_1, δ_1 in (76) substituting q in place of q_1. Table XV will be found convenient in obtaining q and q_1 from $\frac{l}{2}$ and $\frac{l_1}{2}$. The more complicated expressions for the latter are to be used only when it becomes difficult to obtain $1-\sqrt{k'}$ and $1-\sqrt{k}$ without carrying out the calculation of k and k' to an inconvenient number of decimal places.

For relatively long coils, for which the angle $\theta=\tan^{-1}\frac{2a}{b}$ is not greater than 45°, the simple formula

$$K = 1 - \frac{4}{3\pi}\frac{k}{k'} + 2q + 12q^2 + 44q^3 + 116q^4 + 260q^5$$
$$+ 576q^6 + \frac{3760}{3}q^7 + \cdots \qquad [78]$$

will give values of K correct to a few parts in ten million in the most unfavorable case.

The formulas (76), (77), and (78) between them cover the entire range of values of θ with all the precision desired, since the general terms of the series are known. The formula (76) for short coils is the least convenient to use, and for very short coils (69) is preferable. However, by including terms in q^9 in (77) the range of its applicability may be extended to $\theta = 80°$, so that (76) need not be used except as a check.

THE WEBSTER-HAVELOCK FORMULA

Webster[80] in 1905 by the evaluation of a definite integral, involving Bessel functions, derived a formula for the inductance of relatively long solenoids, which is very simple in form. Havelock[81] gives the same formula as a special application of his formulas for the values of certain integrals of Bessel functions, and stated that the first four terms had already been found by Russell,[82] but seems to have been unacquainted with the work of Webster. This formula is

$$L = 4\pi^2\frac{a^2n^2}{b}\left\{1 - \frac{8}{3\pi}\frac{a}{b} + \frac{1}{2}\frac{a^2}{b^2} - \frac{1}{4}\frac{a^4}{b^4} + \frac{5}{16}\frac{a^6}{b^6}\right.$$
$$\left. - \frac{35}{64}\frac{a^8}{b^8} + \frac{147}{128}\frac{a^{10}}{b^{10}} - \cdots\right\} \qquad [79]$$

Both Webster and Havelock gave the same expression for the general term of this series, viz:

$$\frac{(-1)^s(2s)!(2s+2)!}{s!(s+2)!\{(s+1)!\}^2 2^{2s+1}}\left(\frac{a}{b}\right)^{2s+2}$$

80 Bull. of Amer. Math. Soc., **14**, No. 1, p. 1; 1907.

81 Phil. Mag., **15**, p. 332; 1908.

82 Phil. Mag., **13**, eq. (48), p. 445; 1907.

but in all the terms of Webster's final equation (20), except the first two, a factor 2 has been omitted in the denominator of the coefficients.

This expression (79) is in the form adopted by Nagaoka, the expression in the brackets being equivalent to the correction for the ends K tabulated by Nagaoka.

ROSA'S CORRECTION FORMULA

Rosa has shown[83] that the above formulas (69 to 79) apply accurately only to a winding of infinitely thin strip which completely covers the solenoid (the successive turns being supposed to meet at the edges without making electrical contact) and so realizing the uniform distribution of current over the cylindrical surface which has been assumed in the derivation of all the formulas. A winding of insulated wire or of bare wire in a screw thread may have a greater or less self-inductance than that given by the current sheet formulas above according to the ratio of the diameter of the wire to the pitch of the winding. Putting L for the actual self-inductance of a winding and L_s for the current sheet value given by one of the above formulas,

$$L = L_s - \Delta L$$

The correction ΔL is given by the following expression:

$$\Delta L = 4\pi a n\, [A + B] \tag{80}$$

where as above a is the radius, n the whole number of turns of wire and A and B are constants given in Tables VII and VIII, pages 197 and 199.

The correction term A depends on the size of the (bare) wire (of diameter d) as compared with the pitch D of the winding; that is, on the value of the ratio d/D. For values of d/D less than 0.58, A is negative, and in such cases when the numerical values of A are greater than the value of B, which is always positive, the correction ΔL will be negative, and hence L will be *greater* than L_s. (See examples 58 and 63.)

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[83] This Bulletin, 2, pp. 161–187; 1906.

THE SUMMATION FORMULA FOR L[84]

If we have a single layer winding on a cylinder (Fig. 35), the self-inductance is equal to the sum of the self-inductances of the separate turns plus the sum of the mutual inductances of each wire on all the others. Thus, if there are n turns

$$L = nL_1 + 2(n-1)M_{12} + 2(n-2)M_{13} + 2(n-3)M_{14} + \cdot \cdot \cdot + 2M_{1n} \quad [81]$$

where L_1 is the self-inductance of a single turn, M_{12} is the mutual inductance of the first and second turns or any two adjacent turns, M_{13} is the mutual inductance of the first and third or of any two turns separated by one, etc., and M_{1n} is the mutual inductance of the first and last turns. For a coil of four turns this becomes

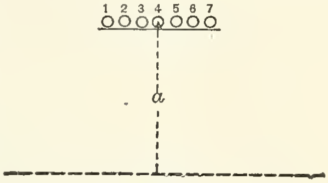

Fig. 35

$$L = 4L_1 + 6M_{12} + 4M_{13} + 2M_{14}$$

L_1 should be calculated by formula (63) or any formula for a circular ring and M_{12}, etc., by (12) or (13). When the number of turns on the coil is small, formula (81) is very convenient, and gives very accurate results.

STRASSER'S FORMULA

Strasser[85] has derived a formula for the self-inductance of a single layer coil of few turns from (81) by substituting for L_1 its value as given by formula (59) and for the various M's their values as given by (12). Strasser's formula with slight correction and some changes in notation is as shown on next page:[86]

---

[84] Kirchhoff, Gesammelte Abhandlungen, p. 177.

[85] Wied. Annal., **17**, p. 763; 1905.

[86] Strasser uses the formula for L as: $L = 4\pi a\left(\log \frac{a}{\rho} + 0.333\right)$. This is not quite correct. It should be

$$L_1 = 4\pi a\left(\log \frac{8a}{\rho} - 1.75\right) = 4\pi a \log\left(\frac{a}{\rho} - 1.75 + \log_e 8\right) = 4\pi a\left(\log \frac{a}{\rho} + 0.32944\right).$$

$$L = 4\pi a \left[n\left(\log \frac{8a}{\rho} - 1.75\right) + n(n-1)\left(\log \frac{8a}{d} - 2\right) - A \right.$$
$$\left. + \frac{d^2}{8a^2}\left\{\left(3 \log \frac{8a}{d} - 1\right)\left(\frac{n^2(n^2-1)}{12}\right) - B\right\}\right] \qquad [82]$$

where n is the whole number of turns, d is the pitch, or distance between the centers of two adjacent turns, a is the mean radius of the coil, ρ is the radius of the section of the wire, and A and B are constants given by Table V, page 195, for values of n up to 30. For coils of a larger number of turns (or indeed any number of turns) the value of L can be accurately calculated by (90) and (93) or by (73) and (80).

SELF-INDUCTANCE OF TOROIDAL COIL OF RECTANGULAR SECTION

The first approximation to the self-inductance of a toroidal coil (that is, a circular solenoid) of rectangular section, wound with a single layer of n turns of wire is

$$L_s = 2n^2 h \log \frac{r_2}{r_1} \qquad [83]$$

where h is the axial depth of the coil, and r_1 and r_2 are the inner and outer radii of the ring, Fig. 36. Formula (83) is exact for a toroidal

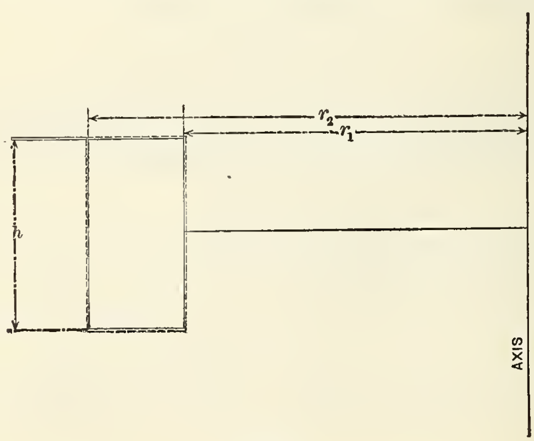

Fig. 36

core enveloped by a current sheet, or for a winding of n turns of infinitely thin tape covering the core completely, the core within

the current sheet being h cm in axial height and $(r_2 - r_1)$ cm in radial breadth.

When the core is wound with round insulated wire, the self-inductance is affected by those lines of force within the cross section of the wire itself, and by those linked with each separate turn of wire in addition to those running through the core. Rosa has shown[87] that the total self-inductance may be more or less than the current sheet value given by (83) according to the size of the wire and the pitch of the winding. In every case, however, the correct value of the self-inductance is derived from the current sheet value L_s by subtracting a correction term ΔL, which is equal to twice the length of the wire multiplied by the sum of two quantities A and B. Thus

$$L = L_s - 2nl(A + B) \qquad [84]$$

where n is the whole number of turns in the winding, l is the length of one turn, A is a quantity, depending on the diameter of the wire and the pitch of the winding, given in Table VII, and B is 0.332. When A is negative and greater than B, L is greater than L_s. This occurs when the pitch of the winding is more than 2.5 times the diameter of the (uncovered) wire.

Fröhlich's formula[88] based on the assumption that a winding of round wires is equivalent to a thick current sheet has been shown to be incorrect.[89]

CHOICE OF FORMULAS

For a coil of only a few turns the summation formula (81), or Strasser's formula (82) give the inductance with great accuracy without the necessity of correction by Tables VII or VIII. Strasser's formula is, however, accurate only for short solenoids, so that the pitch of the winding can not be very great.

For very short solenoids Rayleigh and Niven's formula (69) will give values correct to one in ten thousand for coils whose axial length is as great as one-quarter the diameter of the coil; Coffin's

[87] This Bulletin, 4, p. 141; 1907.

[88] Wied. Annal., 63, p. 142; 1897.

[89] This Bulletin, 4, p. 141; 1907.

extension of this expression (71) gives as great an accuracy for coils as long as one-half the diameter. These two formulas are probably the most convenient for very short solenoids.

For solenoids longer than about one-fifth their diameter the inductance may perhaps most readily be calculated by Nagaoka's formula (75), and the Tables XX and XXI. Havelock's formula (79) is accurate and convenient for coils whose axial length is greater than about one and a quarter times the diameter.

For purposes of great precision, formulas (76), (77), and (78) may be used, (76) being indicated for coils shorter than about one-fifth the diameter, (77) for coils longer than this, and (78) for coils longer than the diameter. Lorenz's absolute formula (73) is of course applicable to coils of all lengths. The interpolation of the elliptic integrals is, however, most easily carried out for coils whose length ranges between one-fifth of the diameter and equality with the latter. The form of this formula is such as to make it necessary in some cases to calculate the separate terms to a greater number of places than are required in the result.

It must be remembered that all these formulas, with the exception of Strasser's and the summation formula (81) give values for a current sheet, and must be corrected to reduce to the actual winding of round wires. This requires the use of formula (80) and Tables VII and VIII.

EXAMPLES ILLUSTRATING THE FORMULAS FOR THE INDUCTANCE OF SINGLE LAYER SOLENOIDS

EXAMPLE 57. RAYLEIGH AND NIVEN'S FORMULA (69) AND CORRECTION FORMULA (80) COMPARED WITH THE SUMMATION FORMULA (81)

$a = 25$ cm, $b = 1$ cm, $n = 10$ turns Fig. 37. Suppose the bare wire is 0.8 mm diameter, the covered wire 1.0 mm.

By formula (69)

$$L_s = 4\pi \times 25 \times 100 \left\{ \log_e 200 - \frac{1}{2} + \frac{1}{20{,}000}\left(\log_e 200 + \frac{1}{4} \right) \right\}$$

$$= 10{,}000\,\pi \times 4.798595$$

$$= 47{,}985.95\,\pi \text{ cm}$$

which is the value of L for a current sheet.

The correction ΔL by formula (80) is $\Delta L = 1000\,\pi\,(A+B)$

Since $D = 1.0$ mm and $d = 0.8$ mm, $d/D = 0.8$

$$\begin{aligned} \text{By Table VII, } A &= 0.3337 \\ \text{" \quad " \quad VIII, } B &= \underline{0.2664} \\ A+B &= 0.6001 \\ \therefore \quad \Delta L &= 600.1\,\pi \text{ cm.} \end{aligned}$$

The value of ΔL calculated to one place more of decimals is $\Delta L = 600.16\,\pi$ cm

$$\therefore \quad L = 47985.95\,\pi - 600.16\,\pi$$

or, $$L = 47385.79\,\pi \text{ cm.}$$

The value of L may also be calculated by the summation formula (81), using Rayleigh and Niven's formula (63) for L_1 and Maxwell's formula (12), for the M's. The following are the values of the ten terms of (81) and the resulting value of L:

$$\begin{aligned} 10\,L_1 &= 6767.196\,\pi \text{ cm} \\ 18\,M_{12} &= 10081.664\,\pi \\ 16\,M_{13} &= 7852.535\,\pi \\ 14\,M_{14} &= 6303.439\,\pi \\ 12\,M_{15} &= 5057.868\,\pi \\ 10\,M_{16} &= 3991.888\,\pi \\ 8\,M_{17} &= 3047.787\,\pi \\ 6\,M_{18} &= 2193.465\,\pi \\ 4\,M_{19} &= 1408.982\,\pi \\ 2\,M_{110} &= \underline{680.982\,\pi} \\ \text{Sum} = L &= 47385.806\,\pi \text{ cm.} \end{aligned}$$

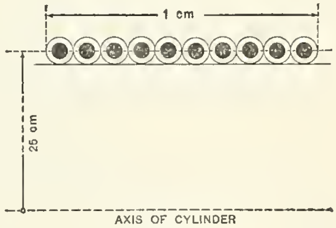

Fig. 37

The difference of less than one in a million between the results obtained by formulas (69) and (80) combined and formula (81) is a good check on the corrections of (80), which amount in this case to more than 1 per cent of the value of the self-inductance. Formula (69) for as short a coil as this is very accurate, the next term, the fourth term of (71), being inappreciable.

If we attempt to use Lorenz's formula in the above example we notice, first, that γ is nearly 89°. The elliptic integrals must consequently be calculated by the series formulas (3), which give their value with all the accuracy desired. We meet, however, with the difficulty that the first and third terms are very nearly equal to one

another and are several hundred times as large as the second term and the sum of the three terms. Consequently, using seven-place logarithms, it is impossible to obtain the self-inductance closer than about five parts in one hundred thousand.

This is also an unfavorable case for (76). Using seven-place logarithms we find

$$K = 21.281755 - 21.220657 = 0.061098$$

and consequently

$$L_s = 47986.27\pi$$

which is about one part in one hundred thousand larger than the correct value.

EXAMPLE 58

As an extreme case to test the use of formulas (69) and (80) we may calculate the self-inductance of a single turn of wire. Let us take the particular case already calculated by Maxwell's and Rayleigh and Niven's formulas (61) and (63), example 52. The radius $a = 25$ cm, the diameter of the bare wire = 1 mm. We may now assume that the wire is covered and that the diameter D is 2 mm. Then $\frac{d}{D} = 0.5$. In using Rayleigh's current sheet formula we take the length of the equivalent current sheet as equal to D. We thus have

$$\begin{aligned} L_s &= 4\pi a \left\{ \log_e \frac{200}{0.2} - \frac{1}{2} + \frac{0.04}{20000} \left(\log_e \frac{200}{0.2} + \frac{1}{4} \right) \right\} \\ &= 100\pi \left\{ 6.907755 - 0.5 + \frac{7.16}{500000} \right\} \\ &= 640.777\pi \text{ cm.} \end{aligned}$$

From Tables VII and VIII $A = -0.1363$ and $B = 0$. Carrying the value of A to one place of decimals more the value is $A = -0.13628$. Thus, since $n = 1$, $\Delta L = 4\pi a\ (-0.13628) = -13.628\pi$, and being negative is added to L_s. Hence

$$\begin{aligned} L &= (640.777 + 13.628)\pi \\ &= 654.405\pi. \end{aligned}$$

This is identical with the value given by the other formulas, example 52.

If we had taken the bare wire of diameter 0.1 cm as equivalent to a current sheet 0.1 cm long in the above formulas for L_s, we should have obtained a different value for L_s, but in that case $\frac{d}{D}$ would be unity and A would be $+.5568$. The resulting value of L would, however, be the same as above.

EXAMPLE 59. COFFIN'S FORMULA (71) COMPARED WITH LORENZ'S (73)

We will use for this case a single layer coil wound on an accurately measured marble cylinder belonging to the Bureau of Standards.

Length of winding, $l = 30.5510$ cm $= b$ in formula (73)
Radius " " $a = 27.0862$ cm
Number of turns $n = 440$

By (71)

$$L_s = 4\pi\overline{440}^2 \times 27.0862 \left\{ 1.4590686 + 0.0878241 - 0.0020427 + .0001651 - 0.0000204 \right\}$$

$$= 4\pi\overline{440}^2 \times 27.0862 \times 1.5449947$$
$$= 101809990 \text{ cm} = 0.10180999 \text{ henry.}$$

By (73)

$$d^2 = 4a^2 + b^2 = 3868.0128$$
$$4a^2 - b^2 = 2001.2858$$
$$\gamma = 60^\circ \; 34' \; 43.'' 655$$
$$\log F = 0.3369388$$
$$\text{"} \; E = 0.0811833$$

Then

$$L_s = \frac{4\pi.\overline{440}^2}{3\,(30.551)^2} \cdot \left\{ 150050.12 + 126105.36 - 158977.00 \right\}$$

or, $L_s = 101810100$ cm $= 0.10181010$ henry.

The correction to be applied to these values is as follows, the diameter of the bare wire being 0.0634 cm, and consequently $\frac{d}{D}$ $=0.9135$:

$$A=0.4664$$
$$B=0.3353$$
$$(A+B)=0.8017$$
$$4\pi na=108.3448\times 440\pi=47671.7\,\pi$$
$$\therefore\ \Delta L=120067 \text{ cm}$$

and

$L=0.10168992$ henry by Coffin's formula
$L=0.10169003$ " by Lorenz's formula.

The agreement between these two formulas is very satisfactory, although in Coffin's formula b is greater than a. For shorter coils the accuracy of this formula is better; for longer coils the error rapidly increases.

EXAMPLE 60. NAGAOKA'S FORMULAS (75) AND (77)

We will take for this the coil in the preceding example

$$a=27.0862 \qquad b=30.5510 \qquad n=440$$

Here $\frac{2a}{b}=1.77318$ and by interpolation in Table XXI using third differences we find

$$K=0.557885-.003165-.000023-.000001$$
$$=0.554696$$

For this case $\theta=60^\circ 34' 43.''655=60.^\circ 57879$, which gives, by interpolation in Table XX,

$$K=0.560382-.005712+.000027-.000001$$
$$=0.554696$$

Substituting this value of K in (75) we find

$$L_s=0.10181013$$

which differs only three parts in ten million from the value found by Lorenz's formula.

Calculating K by (77) we find

$$\sqrt{k'}=0.70087516$$
$$\frac{l}{2}=0.087932623$$

$$q = 0.087943142$$
$$q^2 = 0.007733997$$
$$q^4 = 5.9815 \times 10^{-5}$$
$$q^6 = 4.626 \times 10^{-7}$$

$$\therefore \alpha = 0.00773446 \qquad \gamma = 0.087703884$$
$$\beta = 0.007735385 \qquad 1 - \delta = 0.82423335$$

$$\frac{8\gamma}{(1-\delta)} \cdot \frac{k'^2}{k^2} = 0.27074040$$
$$1 + \frac{8\beta}{1+\alpha} = 1.06140809$$
$$\text{Sum} = 1.33214849$$

multiplied by $\dfrac{2}{3(1-\delta)^2} = 1.3072568$

$$\frac{4}{3\pi}\frac{k}{k'} = 0.7525609$$
$$\therefore K = 0.5546959$$

If we make the calculation with formula (76)

$$1 + \sqrt{k} = 1.93329106$$
$$1 + k = 1.87103210$$
$$\log_{10} k'^2 = \bar{1}.3825629$$
$$\therefore \frac{l_1}{2} = \frac{k'^2}{(1+k)(1+\sqrt{k})^2} = 0.017252700$$

$$q_1 = 0.017252703$$
$$q_1^2 = 0.0002976555$$
$$q_1^4 = 8.86 \times 10^{-8}$$

$$\log_{10} \frac{1}{q_1} = 1.7631429 \qquad \therefore \frac{1}{2} \log_e \frac{1}{q_1} = 2.0298933$$

$$1 + \alpha_1 = 1.00029766 \qquad 1 + \frac{8\beta_1}{1+\alpha_1} = 1.00238054$$

$$\frac{k'^2}{k^2}\left(1 + \frac{8\beta_1}{1+\alpha_1}\right) = 0.31880658$$
$$\frac{8\gamma_1}{1-\delta_1} = 0.14295127$$
$$\text{Sum} = 0.46175785$$
$$\times \frac{1}{2} \log_e \frac{1}{q_1} = 0.93731902$$

$$1-\frac{k'^2}{k^2}=0.68195058$$

$$\text{Sum}=1.61926960$$

multiplied by $\dfrac{1}{3\pi\sqrt{q_1}(1+\alpha_1)^2}=1.3072565$

$$\frac{4}{3\pi}\frac{k}{k'}=0.7525609$$

$$\therefore\ K=0.5546956$$

The two formulas give the same value of K within about one part in two million.

The corresponding values of L_s are:

$$L_s=0.10181010 \text{ by } (77)$$
$$L_s=0.10181005 \text{ `` } (76)$$

the former value being identical with that found by Lorenz's formula. This example illustrates well the advantage of obtaining K from Tables XX and XXI rather than by calculation. The accuracy of these tables is ordinarily more than sufficient.

The correction to be applied to these current sheet values L_s to obtain the self-inductance L, is the same as that calculated in the preceding example.

EXAMPLE 61. WEBSTER-HAVELOCK FORMULA (79) COMPARED WITH NAGAOKA'S FORMULA (78). LONG COIL

$$a=10 \qquad b=40 \qquad N=400$$

and suppose the diameter of the bare wire to be 0.05 cm

$$1+\frac{1}{2}\frac{a^2}{b^2}=\quad 1.03125000$$

$$-\frac{1}{4}\frac{a^4}{b^4}=-0.00097656$$

$$\frac{5}{16}\frac{a^6}{b^6}=\quad 0.00007629$$

$$-\frac{35}{64}\frac{a^8}{b^8}=-0.00000834$$

$$\frac{147}{128}\frac{a^{10}}{b^{10}}=\quad 0.00000110$$

$$-\frac{693}{512}\frac{a^{12}}{b^{12}}=-0.00000008$$

$$\text{Sum}=\quad 1.03034241$$

$$-\frac{8}{3\pi}\frac{a}{b} = -0.21220657$$

$$K = 0.81813584$$

which gives

$$L_s = 0.012919483 \text{ henry}$$

By (78)

$$k^2 = \frac{1}{5} \qquad k'^2 = \frac{4}{5} \qquad \sqrt{k'} = 0.94574152$$

$$1 + k' = 1.89442714$$

$$\therefore \frac{l}{2} = \frac{1}{2}\frac{k^2}{(1+\sqrt{k'})^2(1+k')} = 0.013942859$$

$$\begin{aligned}
q &= 0.013942860 \\
1 + 2q &= 1.02788572 \\
12q^2 &= 0.00233284 \\
44q^3 &= 0.00011926 \\
116q^4 &= 0.00000438 \\
260q^5 &= 0.00000014 \\
576q^6 &= 0.00000000 \\
\text{Sum} &= 1.03034234 \\
\frac{4}{3\pi}\frac{k}{k'} &= 0.21220657 \\
K &= 0.81813577 \\
\therefore L_s &= 0.012919482 \text{ henry}
\end{aligned}$$

which differs by only one part in ten million from the value by the Webster-Havelock formula. The value of K found by interpolation in Nagaoka's tables is $K = 0.818136$.

If we solve this problem by means of Lorenz's formula we are met by the difficulty that $\gamma = 26°$, and therefore the integrals F and E must be taken from Table XII where their values can not be found more accurately than one part in a million.

We find

$$\begin{aligned}
d(4a^2 - b^2)E &= -\ 87909.94 \\
db^2F &= 118752.95 \\
-8a^3 &= -\ 8000.00 \\
\text{Sum} &= 30843.01 \\
\therefore L_s &= 0.01291949 \text{ henry.}
\end{aligned}$$

To find the correction to the current sheet value we have $\frac{d}{D}=0.5$, $n=400$

$$\begin{aligned} A &= -0.1363 \\ B &= +0.3351 \\ \hline A+B &= \quad 0.1988 \\ 4\pi na(A+B) &= \quad 9999 \text{ cm} \\ &= 0.00001000 \text{ henry,} \end{aligned}$$

which must be subtracted from the values of L_s to obtain the self-inductance.

EXAMPLE 62. STRASSER'S FORMULA (82) COMPARED WITH (69) AND (80) AND WITH (81)

Take the coil of 10 turns used in example 57

$$a=25, \qquad d=0.10 \qquad \rho=0.04, \qquad n=10.$$

From Table V, $\quad A=97.9226 \qquad B=4241.59$

Substituting in (82),

$$L=100\pi\left[10\left(\log_e\frac{200}{.04}-1.75\right)+90\left(\log_e\frac{200}{0.1}-2\right)-97.9226 + \frac{0.01}{5000}\left\{\left(3\log_e\frac{200}{0.1}-1\right)\frac{9900}{12}-4241.59\right\}\right]$$

or, $\quad L=100\pi\left[473.8306+0.0275\right]=47385.81\pi$ cm.

This very close agreement with the results by the other two methods (see example 57) is a confirmation of the accuracy of the constants A and B of Table V. Of course, a close agreement with (81) is to be expected, for (82) is derived directly from (81).

EXAMPLE 63. FORMULAS (83) AND (84) FOR TOROIDAL COILS

Professor Fröhlich's standard of self-inductance had the following dimensions:

$r_2=35.05377$ cm = outer mean radius.
$r_1=24.97478$ cm = inner mean radius.
$h=20.08455$ cm = height, center to center of wire.
$\rho=0.011147$ cm = radius of wire.
$n=2738$ = whole number of turns.

These values substituted in (83) give

$$L_s = 0.1020893 \text{ henry.}$$

The correction $\Delta L = -2nl\,(A+B)$ to be substituted in (84) to give the true value of L is found as follows:

The mean spacing of the winding is $D = \pi\dfrac{r_1+r_2}{n} = 0.0689$

The diameter of the bare wire $\qquad d = 2\rho \qquad = 0.0223$

$$\therefore \quad d/D \qquad = 0.324$$

From Table VII,

$$A = -0.572$$
$$B = +0.332^{90}$$
$$\therefore A+B = -0.240$$

$2nl = 2 \times 2738 \times 60.327 = 330300$ cm = whole length of wire in winding.

$$-2nl(A+B) = +79{,}300 \text{ cm}$$
$$= 0.0000793 \text{ henry}$$
$$L_s = 0.1020893 \quad \text{"}$$
$$L = 0.1021686 \quad \text{"}$$

Thus, the correction increases the value of the self-inductance. If the insulation were thinner and the wire thicker (with the same pitch) the correction might be of opposite sign. Thus, if ρ were 0.02 and hence d/D were 0.58, A would be $+0.012$ and ΔL would then be 0.0001130 and $L = 0.1019763$ henry, considerably less than the preceding value.

7. THE SELF-INDUCTANCE OF A CIRCULAR COIL OF RECTANGULAR SECTION

MAXWELL'S APPROXIMATE FORMULA

Maxwell first gave[91] an approximate formula for the important case of a circular coil or conductor of rectangular section, Fig. 38, as follows:

$$L = 4\pi a n^2\left(\log\frac{8a}{R} - 2\right) \qquad [85]$$

[90] This Bulletin, 4, p. 141; 1907. This value applies to any toroidal coils, of 24 turns or more.

[91] Elect. and Mag., Vol. II, § 706.

where R is the geometrical mean distance of the cross section of the coil or conductor. The current is supposed uniformly distributed over this section.

The value of R for any given shape of rectangular section is given by (124). Its value for several particular cases is given in the table on page 168. It is very nearly proportional to the perimeter of the rectangle and approximately equal to $0.2235\ (\alpha+\beta)$ where α and β are the length and breadth of the rectangle.

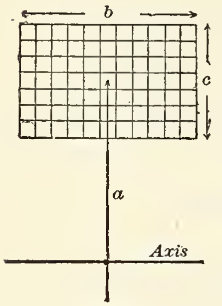

Fig. 38

Formula (85) is derived from (11) by putting R, the geometrical mean distance of the area of the section of the coil from itself, in place of r, the distance between two circles. If we use (12) instead of (11) for this purpose, we shall have a closer approximation to the value of L. Thus,

$$L=4\pi a n^2\left\{\log\frac{8a}{R}\cdot\left(1+\frac{3R^2}{16a^2}\right)-\left(2+\frac{R^2}{16a^2}\right)\right\} \qquad [86]$$

We have placed R^2 in place of d^2 in the second order terms, which is of course not strictly correct, as we should use an arithmetical mean square distance instead of a geometrical mean square distance. (See p. 171.) Nevertheless, (86) is a much closer approximation than (85).

PERRY'S APPROXIMATE FORMULA

Professor Perry has given[92] the following empirical expression for the self-inductance of a short circular coil of rectangular section:

$$L=\frac{4\pi n^2 a^2}{0.2317a+0.44b+0.39c} \qquad [87]$$

in which n is the whole number of turns of wire, a the mean radius, b the axial breadth, c the radial depth. As in all the formulas of this paper, the dimensions are in centimeters and the value of L is in centimeters. This formula gives a good approximation to L as long as b and c are small compared with a.

[92] John Perry, Phil. Mag., 30, p. 223; 1890.

WEINSTEIN'S FORMULA

Maxwell's more accurate expression for the self-inductance of a circular coil of rectangular section[93] was not quite correct. The investigation was repeated by Weinstein,[94] who gave the following formula:

$$L_u = 4\pi a n^2 (\lambda + \mu)$$

where

$$\lambda = \log\frac{8a}{c} + \frac{1}{12} - \frac{\pi x}{3} - \frac{1}{2}\log(1+x^2) + \frac{1}{12x^2}\log(1+x^2)$$
$$+\frac{1}{12}x^2\log\left(1+\frac{1}{x^2}\right) + \frac{2}{3}\left(x - \frac{1}{x}\right)\tan^{-1}x$$
$$\mu = \frac{c^2}{96a^2}\left[\left(\log\frac{8a}{c} - \frac{1}{2}\log(1+x^2)\right)(1+3x^2) + 3.45x^2 + \frac{221}{60}\right.$$
$$\left. - 1.6\pi x^3 + 3.2x^3\tan^{-1}x - \frac{1}{10}\frac{1}{x^2}\log(1+x^2) + \frac{1}{2}x^4\log\left(1+\frac{1}{x^2}\right)\right] \quad [88]$$

b and c are the breadth and depth of the coil and $x = \frac{b}{c}$.

Weinstein's formula for the case of a square section, where $b = c$ reduces to the following simpler expression:

$$L_u = 4\pi a n^2\left\{\left(1 + \frac{c^2}{24a^2}\right)\log\frac{8a}{c} + .03657\frac{c^2}{a^2} - 1.194914\right\} \quad [89]$$

This is a very accurate formula as long as c/a is a small quantity. The current is supposed distributed uniformly over the section of the coil, and hence for a winding of round insulated wire, correction must be made by formula (93).

STEFAN'S FORMULA

Stefan[95] simplified Weinstein's expression (88) by collecting together terms depending on the ratio of b to c and computing two short tables of constants y_1 and y_2. His formula is as follows:

$$L = 4\pi a n^2\left\{\left(1 + \frac{3b^2 + c^2}{96a^2}\right)\log\frac{8a}{\sqrt{b^2 + c^2}} - y_1 + \frac{b^2}{16a^2}y_2\right\} \quad [90]$$

---

[93] Phil. Trans., 1865, and Collected Works.

[94] Wied. Annal., **21**, p. 329; 1884.

[95] Wied. Annal., **22**, p. 113; 1884.

The values of y_1 and y_2 are given in Table VI, page 196, as functions of the ratio of the breadth b to the depth c. The function y_1 is unchanged when b/c and c/b are interchanged. This is not, however, true in the case of the function y_2, which approaches infinity as $x=b/c$ approaches zero. Stefan gave values of y_2 for values of x greater than unity only. These values are reproduced in Table VI, where y_2 is given as a function of c/b, for values between zero and unity; that is, for coils having a breadth greater than their depth. For the case b/c less than unity, values have been calculated and are also included in the table.

For the method of taking the dimensions b and c of the cross section, see page 116, section 6; also example 47, page 97.

LONG COIL OF RECTANGULAR SECTION; I. E., SOLENOID OF MORE THAN ONE LAYER

ROSA'S METHOD

When the coil is so long that the formula of Stefan is no longer accurate, the self-inductance may be accurately calculated by a method given by Rosa.[96]

In Figs. 39, 40, and 41 are shown three coils, having the same length and mean radius. The first is a single winding of thin tape and the self-inductance, calculated by a current sheet formula, is L_s. The second is a single layer of wire of square section (length b, depth c, and b/c turns) and its self-inductance is L_u, the current being supposed uniformly distributed over the area of the square conductors. The third is a winding of round insulated wire of length b, depth c, and any number of layers, and its self-inductance is L. These different self-inductances are related as follows:

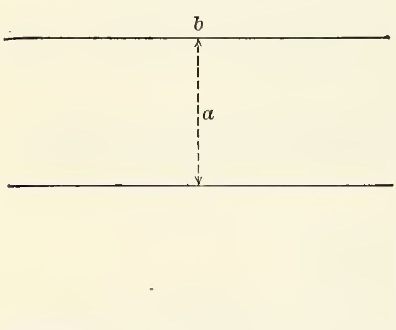

Fig. 39

$$L_s - \varDelta_1 L = L_u$$
$$L_u + \varDelta_2 L = L$$
$$\therefore L = L_s - \varDelta_1 L + \varDelta_2 L$$

[96] This Bulletin, 4, p. 369; 1907.

L_s is calculated by any current sheet formula as (69), (71), (72), or (73). The correction $\Delta_1 L$ for the depth of the coil is given by the following formula:

$$\Delta_1 L = 4\pi a n' \, [A_s + B_s] \tag{91}$$

This formula has the same form as (80), but some of the quantities have a different meaning; a is the mean radius as before, n' is b/c, the number of square conductors in the length b, Fig. 40, and A_s and B_s are given in Tables IX and X.

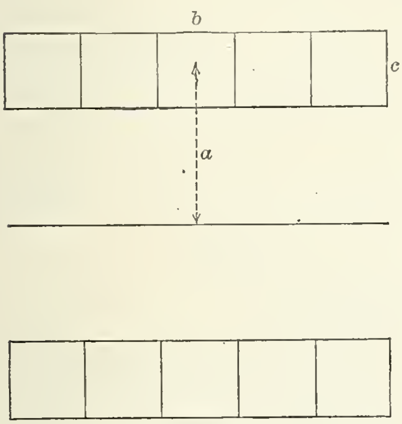

Fig. 40

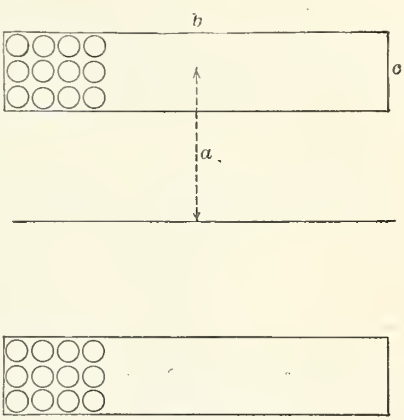

Fig. 41

The correction $\Delta_2 L$ is calculated in precisely the same way as for a short coil, as described below, formula (93). The above formula for $\Delta_1 L$ gives a very accurate value of the correction to be applied to L_s to obtain L_u, and permits a test to be made for the error of Stefan's formula when applied to longer coils than the latter is intended for. Such a calculation shows that for a coil as long as its diameter Stefan's formula (and Weinstein's also, of course) is 1 per cent in error, giving too large a value.

COHEN'S APPROXIMATE FORMULA

Cohen has given the following approximate formula[97] for the self-inductance of a long coil or solenoid of several layers:

[97] This Bulletin, 4, p. 389; 1907.

$$L=4\pi^2n^2m\left\{\frac{2a_0^4+a_0^2l^2}{\sqrt{4a_0^2+l^2}}-\frac{8a_0^3}{3\pi}\right\}$$
$$+8\pi^2n^2\left[\left\{(m-1)a_1^2+(m-2)a_2^2+\cdots\right\}\left(\sqrt{a_1^2+l^2}-\frac{7}{8}a_1\right)\right.$$
$$+\frac{1}{2}\left\{m(m-1)a_1^2+(m-1)(m-2)a_2^2+\cdots\right\}\left(\frac{a_1\delta a}{\sqrt{a_1^2+l^2}}-\delta a\right)$$
$$\left.-\frac{1}{2}\left\{m(m-1)a_1^2+(m-2)(m-3)a_2^2\cdots\right\}\frac{\delta a}{8}\right] \qquad [92]$$

where a_0 is the mean radius of the solenoid, a_1, a_2, $\cdots$ a_m are the mean radii of the various layers in the order of their magnitudes, m is the number of layers and δa is the distance between centers for any two consecutive layers, and n is the number of turns per unit length.

For long solenoids, where the length is, say, four times the diameter, we can neglect the last term in equation (92).

MAXWELL'S CORRECTION FORMULA[98]

GIVING THE VALUE OF Δ_2L

Maxwell has shown that when a coil of rectangular section (Fig. 41) is wound with round insulated wire and the self-inductance is calculated by a formula in which the current is assumed to be distributed uniformly over the section, as in Weinstein's and Stefan's, the calculated value L_u is subject to three corrections, each of which tends to increase the calculated value of the self-inductance. Thus:

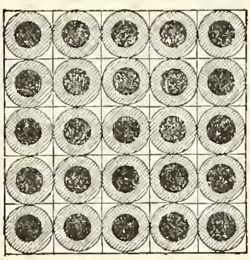

Fig. 42

$$L=L_u+\Delta_2L$$

$$\text{and } \Delta_2L=4\pi an\left\{\log_e\frac{D}{d}+0.13806+E\right\} \qquad [93]$$

Maxwell showed that the first term takes account of the effect of the insulation, d and D being the diameters of the bare and covered wire, respectively, Fig. 42. The second correction term (0.13806)

[98] Elect. and Mag., Vol. II, § 693.

reduces from a square section to a circular section for the conductor. The third correction term E takes account of the differences in the mutual inductances of the separate turns of wire on one another when the wire has a round section from what the mutual inductances would be if the wire were of square section and no space was occupied by insulation. This term was stated by Maxwell to be equal to -0.01971; it was subsequently stated by Stefan to be equal to $+0.01688$. Rosa has shown [99] that its value is variable, depending on the number of turns of wire in the coil and the shape of the cross section of the latter, and has given the values of E for a number of particular cases.

From the following table one can interpolate for E for any particular case not included in the table.

Summary of the values of E found for the various cases considered:

2	turns		$E=$ . . .	0.006528
3	"	(one layer)	$E=$	.009045
4	"	(two layers)	$E=$	.01691
4	"	(one layer)	$E=$	.01035
8	"	(two layers)	$E=$	.01335
10	"	(one layer)	$E=$	.01276
20	"	(one layer)	$E=$	.01357
16	"	(four layers)	$E=$	.01512
100	"	(ten layers)	$E=$	.01713
400	"	(20×20)	$E=$	.01764
1,000	"	(50×20)	$E=$	.01778
Infinite number of turns			$E=$	.01806

The correction $\Delta_2 L$ is much smaller than $\Delta_1 L$, and can be neglected except when the highest accuracy is sought. The value

[99] This Bulletin, 3, p. 37; 1907.

of L_s and $\Delta_1 L$ can be calculated with accuracy if the dimensions are accurately known, and this is possible if one uses enameled wire of uniform section and takes proper care in winding and measuring the coil. However, such a coil can not be recommended for a standard of the highest precision, and the full theory is given for the sake of completeness and to show the magnitude of the smaller corrections, rather than because all the corrections are likely to be generally needed in practice.

CHOICE OF FORMULAS

If the dimensions of the cross section be very small relatively to the mean radius, formula (86) may be used. Formula (85) is a still rougher approximation, as is also (87).

For somewhat larger cross section Weinstein's formula (88) will give good results. Stefan's form (90) of Weinstein's expression is more convenient to use. Formula (89) is convenient and accurate for coils of square cross section. All these formulas assume that the current is uniformly distributed over the cross section of the coil, and must consequently be corrected by formula (93) to reduce to a winding of round wires.

The formulas (88) and (90) begin to be in error for long coils. Cohen's formula (92), however, is most accurate for long solenoids, whose length is more than about four times the diameter.

The most accurate formulas are those of Rosa's method (91) and (93). Since the current sheet value may be very accurately obtained by any of the suitable formulas in section 6, this method may be applied to any solenoidal coil whatever.

EXAMPLES ILLUSTRATING THE FORMULAS FOR THE SELF-INDUCTANCE OF CIRCULAR COILS OF RECTANGULAR SECTION

EXAMPLE 64. MAXWELL'S APPROXIMATE FORMULAS (85), (86) AND PERRY'S APPROXIMATE FORMULA (87) COMPARED WITH WEINSTEIN'S FORMULA (89)

Suppose a coil of mean radius 4 cm, with 100 turns of insulated wire, wound in a square channel 1×1 cm. (Fig. 43.)

Substituting in (85) $a = 4$, $n = 100$, $R = 0.44705$ (the g. m. d. of a square 1 cm on a side) we have

$$L = 4\pi . 4 . \overline{100}^2 \left[\log_e \frac{32}{.44705} - 2 \right]$$

$$= 1.141 \text{ millihenrys.}$$

This is a first approximation to the self-inductance of the coil. Formula (86) gives a second approximation as follows:

$$L = 4\pi . 4 . \overline{100}^2 \left[\log_e \frac{32}{0.44705}\left(1 + \frac{3 \times \overline{0.447}^2}{256}\right) - \left(2 + \frac{\overline{0.447}^2}{256}\right)\right]$$

$$= 1.146 \text{ millihenrys.}$$

Perry's approximate formula, which applies only to relatively short coils, happens to give a very close approximation for this case. Substituting in (87), the above values, and also $b = c = 1$,

$$L = \frac{4\pi \; \overline{100}^2 \times 16}{0.9268 + 0.44 + 0.39}$$

$$= 1.144 \text{ millihenrys.}$$

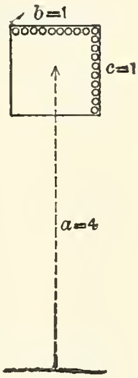

Fig. 43

Substituting in the more accurate formula (89) of Weinstein we shall obtain a value with which to compare the above approximations.

$$L = 160000\pi \left[\left(1 + \frac{1}{384}\right) \log_e \frac{32}{1} + 0.03657 \times \frac{1}{16} - 1.194914 \right]$$

$$= 1.147 \text{ millihenrys.}$$

For $a = 4$, $b = 2$, $c = 1$ $n = 200$

Formula (85)	gives	3.750	millihenrys
" (86)	"	3.787	"
" (87)	"	3.661	"
" (89)	"	3.805	"

For $a = 10$, $b = 1$, $c = 1$, $n = 100$

Formula (85)	gives	4.005	millihenrys
" (86)	"	4.007	"
" (87)	"	3.994	"
" (89)	"	4.008	"

It will be seen that formula (87) does not give as close approximations as the others, except in the case of the first example, where it

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happens to give a value very close to that given by (89). All the values, those of (89) included, are subject to correction by (93) when the coil is wound with round insulated wire.

EXAMPLE 65. FORMULAS (89) AND (90) COMPARED WITH CURRENT-SHEET FORMULAS

As a test of these formulas we may calculate the self-inductance of a single turn of wire, using the case already calculated in example 52; that is, a circle of radius $a=25$ cm, and the diameter of the bare wire is 1 mm. Substituting these values in (89) we have

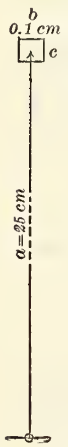

Fig. 44

$$L=100\pi\left[\left(1+\frac{.01}{15000}\right)\log_e 2000+\frac{.03657}{(250)^2}-1.194914\right]$$

$$=640.5995\ \pi \text{ cm.}$$

Substituting in (90),

$$L=100\pi\left[\left(1+\frac{.01}{15000}\right)\log_e \frac{200}{\sqrt{.02}}-0.848340+\frac{.01\times .8162}{10000}\right]$$

$$=640.5995\ \pi \text{ cm,}$$

agreeing with the value by (89).

These values are for a conductor of square cross section (Fig. 44). To reduce to a circular section of same diameter (0.1 cm) we must apply the second correction term of (93); that is, add to the above value

$$\Delta L=4\pi a\times 0.138060$$

$$\text{Thus, } L=(640.5995+13.8060)\ \pi$$

$$=654.4055\pi \text{ cm,}$$

which agrees with the value found for the self-inductance of a round wire 0.1 cm diameter, bent into a circle of 25 cm radius, by formula (63) example 52 and formulas (69) and (80), example 58.

EXAMPLE 66. STEFAN'S FORMULA (90) COMPARED WITH (69) BY MEANS OF ROSA'S CORRECTION FORMULA (91)

Suppose a coil of mean radius 10 cm, wound with 100 turns in a square channel 1×1 cm. Assuming the current uniformly distributed we obtain from (90), in which $y_1=0.848340$, $y_2=0.8162$,

$$\log_e \frac{8a}{\sqrt{b^2+c^2}} = \log_e \frac{80}{\sqrt{2}} = 4.03545$$

$$L_u = 4\pi \times 100{,}000 \left[\left(1 + \frac{4}{9600} \right) 4.03545 - 0.84834 + 0.00051 \right]$$

$$= 4\pi \times 318{,}930 \text{ cm}$$
$$= 4.00779 \text{ millihenrys.}$$

By formula (69) we have for the self-inductance of a current sheet for which $a=10$, $b=1$, $n=1$,

$$L_s = 4\pi \times 38.83475 \text{ cm.}$$

This is larger than the value for the coil of section 1×1 by $\varDelta_1 L$, the value of the latter being given by formula (91).

By Table IX, $A_s = 0.6942$. More closely, it is 0.69415.[100]

By Table X, $B_s = 0$. In this case $n' = 1$. Hence,

$$\varDelta_1 L = 4\pi \times 10 \times 0.69415 = 4\pi \times 6.9415 \text{ cm}$$
$$\therefore L_1 = 4\pi \; (38.83475 - 6.9415) = 400.782 \text{ cm.}$$

This is the value of the self-inductance for one turn only, the current being uniformly distributed. For 100 turns L is 10^4 times as great.

$$\therefore L_u = 4.00782 \text{ millihenrys.}$$

This value agrees with the above value by Stefan's formula within less than one part in one hundred thousand.

For a coil of insulated round wires, this result must be corrected by formula (93).

For a coil of the same radius, but of length $b=10$ cm, $c=1$ cm, wound with 10 layers of 100 turns each, we have the following values:

By Stefan's formula, $y_1 = 0.59243$, $y_2 = 0.1325$

$$L_u = 4\pi \times 10 \times \overline{1000}^2 \times 1.55536$$
$$= 195.452 \text{ millihenrys.}$$

[100] This Bulletin, 4, p. 369; 1907.

By (69) the current sheet value of L for 10 turns is

$$\begin{aligned} L_{10} &= 4\pi \times 10 \times 100 \times 1.65095 \\ &= 4\pi \times 1650.95. \end{aligned}$$

The correction for depth of section by (91) is, since by Tables IX and X, $A_s = 0.6942$, $B_s = 0.2792$, and therefore $A_s + B_s = 0.9734$

$$\begin{aligned} \varDelta_1 L &= 4\pi 10 \times 10 \times 0.9734 \\ &= 4\pi \times 97.34 \\ \therefore L_u = L_{10} - \varDelta_1 L &= 4\pi(1650.95 - 97.34) \\ &= 4\pi \times 1553.61 \text{ cm for 10 turns.} \end{aligned}$$

For $n = 1000$ turns the self-inductance will be $\overline{100}^2$ times as great.

$$\begin{aligned} L_u &= 4\pi \times 15.5361 \times 10^6 \text{ cm} \\ &= 195.232 \text{ millihenrys.} \end{aligned}$$

This value is about 1 part in 900 smaller than the above value, showing that Stefan's formula gives too large results by that amount for a coil of this length. If the coil were twice as long, the error would be about ten times as great.

It is interesting to obtain by this method an estimate of the error by Stefan's formula for coils longer than those for which it is intended. For short coils it is seen to be very accurate, subject always to the corrections of formula (93), and for longer coils it gives a good approximation. The method of (91), however, applies to coils of any length.

EXAMPLE 67. STEFAN'S FORMULA (90) COMPARED WITH (81) AND WITH STRASSER'S (82) FOR COILS OF FEW TURNS, USING THE CORRECTION FORMULA (93)

Coil of 2 turns of wire, 0.4 mm diameter, wound in a circle of 1.46 cm radius with a pitch of 2 mm. Stefan's formula assumes a uniform distribution over a rectangular section. Suppose a section as shown in Fig. 45, 4×2 mm, with one turn of wire in the center of each square. For the rectangular section, with the current uniformly distributed, the self-inductance by Stefan's formula is with $a = 1.46$, $c/b = 0.5$, $y_1 = 0.7960$, $y_2 = 0.3066$, $L_u = 4\pi a n^2 \times 2.4763 = 4\pi a n \times 4.9526$, n being 2. To reduce this to the case of a

winding of 2 turns of wire as shown we must apply the corrections given by (93) thus:

$$\begin{aligned}
\log D/d = \log_e 5 &= 1.60944 \\
\text{second term} &= 0.13806 \\
\text{third term } E &= 0.00653 \\
\hline
&\quad 1.7540
\end{aligned}$$

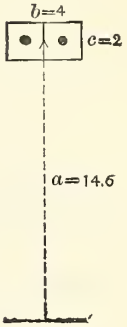

Fig. 45

$$\therefore \Delta_2 L = 4\pi a n \times 1.7540$$

$$L = L_u + \Delta_2 L = 4\pi a n \times 6.7066 = 246.1 \text{ cm.}$$

By the summation formula (81) we have in this case

$$\begin{aligned}
L &= 2L_1 + 2M_{12} \\
&= 4\pi a[9.2400 + 4.1606] \\
&= 245.86 \text{ cm.}
\end{aligned}$$

The value by Strasser's formula is the same as by the summation formula to which it is equivalent. We have also used formulas (69) and (80) for this case and have obtained 246.0.

This is one of several problems calculated by Drude[101] by Stefan's formula. Drude concluded that Stefan's formula was inapplicable to such coils, as it gave results from 10 to 25 per cent too large. His trouble was, however, due to taking the length of the coil as the distance between the center of the first wire and the center of the last (instead of n times the pitch) and neglecting the correction terms of formula (93). As we have seen above, Stefan's formula when properly used can be depended upon to give accurate results for short coils, and results within less than 1 per cent for coils of length equal to the radius of the coil.

We have calculated several other cases given by Drude and give below the results, together with his experimental values. The radius is the same in each case, and the numbers in the first column are the number of turns in the several coils.

[101] Wied. Annal., 9, p. 601; 1902.

n	By Stefan's Formula (90) and (93)	By Rayleigh's Formula (69) and (80)	By Strasser's Formula (82) or (81)	Drude's Observed Values (Values of L in Centimeters)
2	246.1	246.0	245.9	238.5
4	711.9	711.1	710.8	697.9
6	1298.7	1297.7	1297.8	1271.4
9	2318.0	2313.0	2315.7	2300.1

It will be seen that the values by the different formulas agree very closely, and that the experimental values agree as closely as could be expected for such small inductances.

EXAMPLE 68. FORMULAS (69) AND (80) COMPARED WITH (90) AND (93) FOR COIL OF 20 TURNS WOUND WITH A SINGLE LAYER

$$a = 25 \qquad b = 2 \text{ cm} \qquad c = 0.1 \text{ cm} \qquad n = 20.$$

Diameter of bare wire 0.6 mm, of covered wire 1.0 mm.

In the last case we obtained the self-inductance of the coil by two distinct methods, the first being the method of summation, the second by assuming the current uniformly distributed over the section, and then applying the three corrections C, F, E. In this problem we may first calculate L by use of the current sheet formula (69), and then apply the corrections for section, A and B formula (80); and, second, by Stefan's formula for uniform distribution, and apply the three corrections C, F, E, which give the value for a winding of round insulated wires.

Rayleigh's formula for this example gives:

$$L = 4\pi a n^2 \left\{ \log_e 100 - 0.5 + \frac{4}{20{,}000}\left(\log_e 100 + \frac{1}{4} \right) \right\}$$

$$\log_e 100 = 4.605170$$

$$\frac{4}{20{,}000}\left(\log_e 100 + \frac{1}{4} \right) = \underline{0.000971}$$

$$4.606141$$

$$\underline{-0.500000}$$

$$4.106141$$

$$4\pi a n^2 = 40{,}000\pi \qquad \therefore L_s = 164\,245.64\pi \text{ cm.}$$

This is the self-inductance of a winding of 20 turns of infinitely thin tape, each turn being 1 mm wide, with edges touching without

making electrical contact, which arrangement fulfills the conditions of a current sheet. To reduce this to the case of round wires we must apply the corrections A and B for self and mutual induction.[102]

$$
\begin{aligned}
&\text{By Table VII, for } d/D=0.6, & A&=0.0460\\
&\text{By Table VIII, for } n=20, & B&=\underline{0.2974}\\
& & A+B&=0.3434\\
&4\pi an= & &2{,}000\pi\\
&\Delta L=4\pi an(A+B)= & &686.8\pi \text{ cm}\\
&L=L_s-\Delta L= & &163\ 558.84\pi \text{ cm.}
\end{aligned}
$$

By Stefan's formula we find, sustituting the above values of a, n, b, c, and taking $y_1=0.548990$ and $y_2=0.1269$

$$L_u=162\ 234.60\pi \text{ cm.}$$

The correction E for a single layer coil of 20 turns is given on page 141. The three corrections are then as follows:

$$
\begin{aligned}
C&=0.13806\\
F&=0.51082=\log_e\frac{10}{6}\\
E&=\underline{0.01357}\\
\text{Sum}&=0.66245.
\end{aligned}
$$

$$\therefore \Delta L=4\pi an(C+F+E)=1324.90\pi \text{ cm.}$$

$$\therefore L=L_u+\Delta L=163\ 559.50\pi \text{ cm.}$$

This value of L is greater than the value found by the other method by only four parts in a million. Thus we see that the method of calculating L_u by Stefan's or Weinstein's formula and applying the corrections C, F, E gives practically identical results with the method of summation and also with the current sheet method *for short coils.* When, however, the coils are longer, the agreement is not so good, for the reason that the formula of Weinstein (and Stefan's, derived from it) is not as accurate when the section of the coil is greater. Thus if the coil in the above problem had been 5 cm long and 2.5 mm deep and wound with 20 turns of heavier wire, the difference would have been one part in twenty-five thousand (still very good agreement), and if it were 10 cm long and

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[102] Rosa, this Bulletin, 2, p. 161; 1906.

0.5 cm deep (the radius being 25 cm) it would have been one part in two thousand two hundred. For most experimental work, therefore, Stefan's formula is amply accurate.

EXAMPLE 69. COHEN'S FORMULA (92) COMPARED WITH (91)

A solenoid of length $l=50$ cm, mean radius 5 cm, depth of winding 0.4 cm, is wound with 4 layers of wire of 500 turns each. Substituting these values in (92) we have ($n=10$)

$$\begin{aligned} L_s &= 16\pi^2 n^2(1144.3+3336.0-10.84-1.04) \\ &= 70.562 \text{ millihenrys.} \end{aligned}$$

By the second method we first find L_s by (69), then $\Delta_1 L$ by (91), and $\Delta_2 L$ by (93)

$$\begin{aligned} L_s &= 72.648 \text{ millihenrys} \\ -\Delta_1 L &= -2.167 \quad \text{"} \\ \Delta_2 L &= \underline{0.048} \quad \text{"} \\ L &= 70.529 \quad \text{"} \end{aligned}$$

This shows a very close agreement between (92) and (91).

In calculating L_s we may use Table IV. Since $d/l=0.2$

$$Q=3.6324, \quad an^2=5\times\overline{2000}^2=20{,}000{,}000$$

$$L_s=3.6324\times 20{,}000{,}000 \text{ cm}$$

or,

$$L_s=72.648 \text{ millihenrys.}$$

8. SELF AND MUTUAL INDUCTANCE OF LINEAR CONDUCTORS [103]

SELF-INDUCTANCE OF A STRAIGHT CYLINDRICAL WIRE

The self-inductance of a length l of straight cylindrical wire of radius ρ is

$$L=2\left[l\log\frac{l+\sqrt{l^2+\rho^2}}{\rho}-\sqrt{l^2+\rho^2}+\frac{l}{4}+\rho\right] \qquad [94]$$

$$=2l\left[\log\frac{2l}{\rho}-\frac{3}{4}\right] \text{ approximately.} \qquad [95]$$

Where the permeability of the wire is μ, and that of the medium outside is unity, (95) appears in the form

$$L=2l\left[\log\frac{2l}{\rho}-1+\frac{\mu}{4}\right] \qquad [96]$$

[103] See paper by E. B. Rosa, this Bulletin, 4, p. 301; 1907.

This formula was originally given by Neumann.

For a straight cylindrical tube of infinitesimal thickness, or for alternating currents of great frequency, when there is no magnetic field within the wire, the self-inductance is

$$L = 2l\left[\log\frac{2l}{\rho} - 1\right] \qquad [97]$$

This is obtained by subtracting from (95) $l/2$ or from (96) $\mu\, l/2$, the magnetic flux within the conductor due to unit current.

THE MUTUAL INDUCTANCE OF TWO PARALLEL WIRES

The mutual inductance of two parallel wires of length l, radius ρ, and distance apart d is the number of lines of force, due to unit current in one, which cut the other when the current disappears.

This is

$$M = 2\left[l\log\frac{l+\sqrt{l^2+d^2}}{d} - \sqrt{l^2+d^2} + d\right] \qquad [98]$$

$$\therefore M = 2l\left[\log\frac{2l}{d} - 1 + \frac{d}{l}\right] \text{ approximately} \qquad [99]$$

when the length l is great in comparison with d.

Equation (98), which is an exact expression when the wires have no appreciable cross section, is not an exact expression for the mutual inductance of two parallel cylindrical wires, but is not appreciably in error even when the section is large and d is small if l is great compared with d.

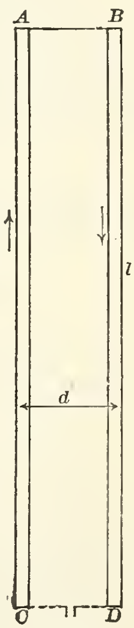

Fig. 46

THE SELF-INDUCTANCE OF A RETURN CIRCUIT

If we have a return circuit of two parallel wires each of length l (the current then flowing in opposite direction in the two wires) the self-inductance of the circuit, neglecting the effect of the end connections shown by dotted lines, Fig. 46, will be very approximately

$$L = 4l\left[\log\frac{d}{\rho} + \frac{\mu}{4} - \frac{d}{l}\right] \qquad [100]$$

In the usual case of $\mu = 1$ this will be, when d/l is small

$$L = 4l\left[\log \frac{d}{\rho} + \frac{1}{4}\right] \qquad [101]$$

If the end effect is large, as when the wires are relatively far apart, use the expression for the self-inductance of a rectangle below (107); or, better, add to the value of (100) the self-inductance of AB + CD, using equation (94) in which $l = 2$AB.

Experimental work at the Bureau of Standards, not yet published, has shown that formula (100), and therefore (94) and (98) are consistent with the formula (63) for the inductance of a circular ring.

[This is equivalent to the following formula in which the logarithms are common:

$$L = 0.7411 \log_{10} \frac{d}{\rho} + .0805 \text{ in millihenrys per mile of conductor,}$$

$$= 0.4605 \log_{10} \frac{d}{\rho} + .050 \text{ in millihenrys per kilometer of conductor,}$$

d and ρ being expressed in centimeters, inches, or any other unit.]

MUTUAL INDUCTANCE OF TWO LINEAR CONDUCTORS IN THE SAME STRAIGHT LINE

The mutual inductance of two adjacent linear conductors of lengths l and m in the same straight line is

$$M_{lm} = l \log \frac{l+m}{l} + m \log \frac{l+m}{m}, \text{ approximately.} \qquad [102]$$

This approximation is very close indeed if the radius of the conductor (which has been assumed zero) is very small.

THE SELF-INDUCTANCE OF A STRAIGHT RECTANGULAR BAR

The self-inductance of a straight bar of rectangular section is, to within the accuracy of the approximate formula (99), the same as the mutual inductance of two parallel straight filaments of the same length separated by a distance equal to the geometrical mean distance of the cross section of the bar. Thus,

$$L = 2l\left[\log \frac{2l}{R} - 1 + \frac{R}{l}\right] \qquad [103]$$

where R is the geometrical mean distance of the cross section of the rod or bar. If the section is a square, $R=0.447\ \alpha$, α being the side of the square. If the section is a rectangle, the value of R is given by Maxwell's formula (124).

This is equivalent to the following:

$$L=2l\left[\log\frac{2l}{\alpha+\beta}+\frac{1}{2}+\frac{0.2235(\alpha+\beta)}{l}\right] \qquad [104]$$

In the above formula L is the self-inductance of a straight bar or wire of length l and having a rectangular section of length α and breadth β.

TWO PARALLEL BARS. SELF AND MUTUAL INDUCTANCE

The mutual inductance of two parallel straight, square, or rectangular bars is equal to the mutual inductance of two parallel wires or filaments of the same length and at a distance apart equal to the geometrical mean distance of the two areas from one another. This is very nearly equal in the case of square sections to the distance between their centers for all distances, the g. m. d. being a very little

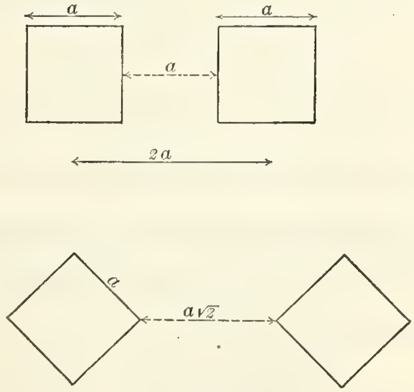

Fig. 47

greater for parallel squares, and a very little less for diagonal squares[104] (Fig. 47). We should, therefore, use equation (99) with d equal to g. m. d. of the sections from one another; that is, substantially, to the distances between the centers.

[104] Rosa, this Bulletin, 3, p. 1; 1907.

The self-inductance of a return circuit of two such parallel bars is equal to twice the self-inductance of one minus twice their mutual inductance. That is,

$$L = 2[L_1 - M]$$

in which L_1 is calculated by (104) and M by (99).

SELF-INDUCTANCE OF A SQUARE

The self-inductance of a square may be derived from the expressions for the self and mutual inductance of finite straight wires from the consideration that the self-inductance of the square is the sum of the self-inductances of the four sides minus the mutual inductances. That is,

$$L = 4L_1 - 4M$$

the mutual inductance of two mutually perpendicular sides being zero. Substituting a for l and d in formulas (94) and (98) we have,

neglecting ρ^2/a^2, $L = 8a(\log \frac{a}{\rho} + \frac{\rho}{a} - .524)$ [105]

where a is the length of one side of the square and ρ is the radius of the wire. If we put $l = 4a =$ whole length of wire in the square,

$$L = 2l\left(\log \frac{l}{\rho} + \frac{4\rho}{l} - 1.910\right)$$

or, $L = 2l\left(\log \frac{l}{\rho} - 1.910\right)$ approximately. [106]

Formulas (105) and (106) were first given by Kirchhoff[105] in 1864.

SELF-INDUCTANCE OF A RECTANGLE

(*a*) *The conductor having a circular section*

The self-inductance of the rectangle of length a and breadth b is

$$L = 2(L_a + L_b - M_a - M_b)$$

where L_a and L_b are the self-inductances of the two sides of length a and b taken alone, M_a and M_b are the mutual inductances of the two opposite pairs of length a and b, respectively.

From (94) and (98) we therefore have, neglecting ρ^2/a^2, and putting d for the diagonal of the rectangle $= \sqrt{a^2 + b^2}$

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[105] Gesammelte Abhandlungen, p. 176. Pogg. Annal., **121**, 1864.

$$L=4\left[(a+b)\log\frac{2ab}{\rho}-a\log(a+d)-b\log(b+d)\right.$$
$$\left.-\frac{7}{4}(a+b)+2(d+\rho)\right]. \qquad [107]$$

(*b*) *The conductor having a rectangular section*

For a rectangle made up of a conductor of rectangular section $\alpha\times\beta$,

$$L=4\left[(a+b)\log\frac{2ab}{\alpha+\beta}-a\log(a+d)-b\log(b+d)\right.$$
$$\left.-\frac{a+b}{2}+2d+0.447(\alpha+\beta)\right] \qquad [108]$$

where as before d is the diagonal of the square. This is equivalent to Sumec's exact formula [106] (6*a*).

For $a=b$, a square,

$$L=8a\left[\log\frac{a}{\alpha+\beta}+0.2235\frac{\alpha+\beta}{a}+0.726\right] \qquad [109]$$

If $\alpha=\beta$, that is, the section of the conductor is a square,

$$L=8a\left[\log\frac{a}{\alpha}+0.447\frac{\alpha}{a}+0.033\right] \qquad [110]$$

MUTUAL INDUCTANCE OF TWO EQUAL PARALLEL RECTANGLES

For two equal parallel rectangles of sides a and b and distance apart d the mutual inductance, which is the sum of the several mutual inductances of parallel sides, is,

$$M=4\left[a\log\left(\frac{a+\sqrt{a^2+d^2}}{a+\sqrt{a^2+b^2+d^2}}\cdot\frac{\sqrt{b^2+d^2}}{d}\right)\right.$$
$$\left.+b\log\left(\frac{b+\sqrt{b^2+d^2}}{b+\sqrt{a^2+b^2+d^2}}\cdot\frac{\sqrt{a^2+d^2}}{d}\right)\right]$$
$$+8\left[\sqrt{a^2+b^2+d^2}-\sqrt{a^2+d^2}-\sqrt{b^2+d^2}+d\right] \qquad [111]$$

---

[106] Elektrotech. Zs., 27, p. 1175; 1906.

For a square, where $a=b$, we have

$$M=8\left[a \log\left(\frac{a+\sqrt{a^2+d^2}}{a+\sqrt{2a^2+d^2}}\cdot\frac{\sqrt{a^2+d^2}}{d}\right)\right]$$
$$+8\left[\sqrt{2a^2+d^2}-2\sqrt{a^2+d^2}+d\right] \qquad [112]$$

Formula (111) was first given by F. E. Neumann[107] in 1845.

The case of two rectangles symmetrically placed about a common vertical axis, the horizontal sides of the smaller rectangle being equidistant from those of the larger rectangle, has been discussed by Martens[108] and a formula derived which enables the mutual inductance to be found for any angle ξ between the planes of the rectangles. This formula is, however, very elaborate and calculations therewith laborious.

SELF AND MUTUAL INDUCTANCE OF THIN TAPES

The self-inductance of a straight, thin tape of length l and breadth b (and of negligible thickness), Fig. 48 (1), is equal to the mutual inductance of two parallel lines of distance apart R_1 equal to the geometrical mean distance of the section, which is $0.22313b$, or

$\log R_1 = \log b - \frac{3}{2}.$

Thus we have approximately

$$L=2l\left[\log\frac{2l}{R_1}-1\right]$$
$$=2l\left[\log\frac{2l}{b}+\frac{1}{2}\right] \qquad [113]$$

If the thickness of the tape is not negligible, this formula becomes, when a is the thickness of the tape,

$$L=2l\left[\log\frac{2l}{b}-\frac{a}{b}+\frac{1}{2}\right] \qquad [114]$$

A closer approximation to L is given by (104), in which α is the thickness and β is the breadth of the tape. For two such tapes in the same plane, coming together at their edges with-

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[107] Allgemeine Gesetze der Inducirten Ströme, Abh. Berlin Akad.

[108] Ann. der Phys. **29**, p. 963; 1909.

out making electrical contact, Fig. 48 (2), the mutual inductance is

$$M = 2l\left[\log\frac{2l}{R_2} - 1\right]$$
$$= 2l\left[\log\frac{2l}{b} - 0.8863\right] \qquad [115]$$

where R_2 is the geometrical mean distance of one tape from the other, which in this case is $0.89252b$. For a return circuit made up of these two tapes the self-inductance is

$$L = 2L_1 - 2M$$
$$= 4l\left(\log\frac{R_2}{R_1}\right) = 4l\log_e 4 \qquad [116]$$
$$= 5.545 \times \textit{length of one tape.}$$

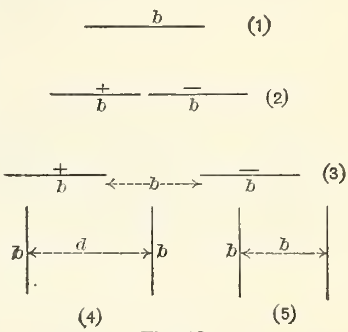

Fig. 48

Thus the self-inductance of such a circuit is independent of the width of the tapes. If the tapes are separated by the distance b, Fig. 48 (3), equal to the width of the tapes, $R_2 = 1.95653b$ and $L = 8.685l$.

If the two tapes are not in the same plane, but parallel, Fig. 48 (4),

$$L = 2L_1 - 2M = 4l\log\frac{R_2}{R_1} \qquad [117]$$

and when the distance apart is equal to the breadth of the tapes, Fig. 48 (5), we have

$$\log\frac{R_2}{R_1} = \frac{\pi}{2}$$

and

$$L = 4l\frac{\pi}{2} = 2\pi l \qquad [118]$$

In this case, also, the self-inductance [2π cm per unit of length] of the pair of thin strips is independent of their width so long as the distance apart is equal to their width. Formula (117) with (132)

may be employed to calculate the self-inductance of a noninductive shunt made up of a sheet of thin metal doubled on itself.

CONCENTRIC CONDUCTORS

The self-inductance of a thin, straight tube of length l and radius a_2, when a_2/l is very small, is given by (97),

$$L_2 = 2l\left[\log \frac{2l}{a_2} - 1\right]$$

The mutual inductance of such a tube on a conductor within it is equal to its self-inductance, since all the lines of force due to the outer tube cut through the inner when they collapse on the cessation of current. The self-inductance of the inner conductor, supposed a solid cylinder, is

$$L_1 = 2l\left[\log \frac{2l}{a_1} - \frac{3}{4}\right]$$

If the current goes through the latter and returns through the outer tube, the self-inductance of the circuit is

$$L = L_1 + L_2 - 2M = L_1 - L_2$$

since M equals L_2

$$\therefore\ L = 2l\left[\log \frac{a_2}{a_1} + \frac{1}{4}\right] \qquad [119]$$

This result can also be obtained by integrating the expression for the force outside a_1 between the limits a_1 and a_2, and adding the term for the field within a_1, there being no magnetic field outside a_2.

If the outer tube has a thickness $a_3 - a_2$ and the current is distributed uniformly over its cross section the self-inductance will be a little greater, the geometrical mean distance from a_1 to the tube, which is more than a_2 and less than a_3, being given by the expression

$$\log a_g = \frac{a_3^2 \log a_3 - a_2^2 \log a_2}{a_3^2 - a_2^2} - \frac{1}{2}$$

Putting this value of log a in (119) in place of log a_2, we should have the self-inductance of the return circuit.

If the current is alternating and of very high frequency, the current would flow on the outer surface of a_1 and on the inner surface

of the tube, and L for the circuit would be

$$L = 2l \log \frac{a_2}{a_1} \qquad [120]$$

MULTIPLE CONDUCTORS

If a current be divided equally between two wires of length l, radius ρ and distance d apart, the self-inductance of the divided conductor is the sum of their separate self-inductances plus twice their mutual inductance.

Thus, when d/l is small,

$$L = 2l\left[\log \frac{2l}{(\rho d)^{\frac{1}{2}}} - \frac{7}{8}\right] = 2l\left[\log \frac{2l}{(r_g d)^{\frac{1}{2}}} - 1\right] \qquad [121]$$

where r_g, the g. m. d. of the section of the wire is 0.7788ρ for a round section.

If there are three straight conductors in parallel and distance d apart, the self-inductance is similarly

$$L = 2l\left[\log \frac{2l}{(r_g d^2)^{\frac{1}{3}}} - 1\right] \qquad [122]$$

The expression $(r_g d^2)^{\frac{1}{3}}$ is the g. m. d. of the multiple conductor.

EXAMPLES ILLUSTRATING THE FORMULAS FOR THE SELF AND MUTUAL INDUCTANCE OF LINEAR CONDUCTORS

EXAMPLE 70. FORMULAS (94), (95), (96), AND (97)

A straight copper wire 100 cm long and 0.2 cm diameter will have a self-inductance by formula (95) of

$$L = 200\left(\log_e \frac{200}{0.1} - \frac{3}{4}\right) = 1370.18 \text{ cm.}$$

If it were twice as long

$$L = 400\left(\log_e \frac{400}{0.1} - \frac{3}{4}\right) = 3017.62 \text{ cm.}$$

The more exact formula (94) gives practically the same result where ρ is so small compared with l.

If the wire were of iron with a permeability of 1000, we should have in the first case for $l = 100$

$$L = 200\ (\log_e 2000 - 1 + 250) = 51320 \text{ cm.}$$

For sufficiently rapid oscillations so that the current may be considered to be confined to the surface of the wire

$$L = 200\ (\log_e 2000 - 1) \ = 1320.18 \text{ cm.}$$

If the length of the conductor were 10 meters and the diameter 0.2 cm as before, the self-inductance by (95) would be

$$L = 2000\left(\log_e 20000 - \frac{3}{4}\right) = 18307.0 \text{ cm}$$

$$= 18.307 \text{ microhenrys.}$$

EXAMPLE 71. FORMULAS (98) AND (99)

Two parallel copper wires of length 100 cm and distance apart 200 cm will have a mutual inductance of

$$M = 2\left[100 \log_e \frac{100 + 100\sqrt{5}}{200} - 100\sqrt{5} + 200\right]$$

$$= 200\left[\log_e \frac{1 + \sqrt{5}}{2} - \sqrt{5} + 2\right]$$

$$= 200\ (\log_e 1.61803 - 0.2361)$$
$$= 49.02 \text{ cm.}$$

If the length of each conductor were 200 cm and the distance apart 100 cm, then

$$M = 400\left[\log_e \frac{2 + \sqrt{5}}{1} - \frac{\sqrt{5}}{2} + \frac{1}{2}\right] = 330.24 \text{ cm.}$$

The approximate formula (99) is only applicable when the length of the conductors is great compared with their distance apart. Suppose two conductors 10 meters long are 10 cm apart, then by (99)

$$M = 2000\left[\log_e \frac{2000}{10} - 1 + \frac{10}{1000}\right]$$
$$= 2000\,[5.2983 - 0.9900]$$
$$= 8616.6 \text{ cm} = 8.6166 \text{ microhenrys.}$$

The formula (98) gives a value less than two parts in one hundred thousand greater.

EXAMPLE 72. FORMULAS (100) AND (101)

Suppose a return circuit of two parallel wires, each 10 meters long and 0.2 cm diameter, distant apart 10 cm, center to center, Fig. 49. The self-inductance of the circuit, neglecting the ends, is by (100)

$$L = 4000\left[\log_e \frac{10}{0.1} + \frac{1}{4} - \frac{10}{1000}\right]$$
$$= 4000 \times 4.8452$$
$$= 19380.8 \text{ cm} = 19.3808 \text{ microhenrys.}$$

Fig. 49

We have already calculated (example 70) the self-inductance of one of these two wires by itself. Doubling the value we have 36.6140 microhenrys as the self-inductance of two wires in series. In example 71 we calculated the mutual inductance of these two wires. Doubling the value for M we have 17.2332 microhenrys. The resultant self-inductance of the circuit (neglecting the ends) is

$$L = 2L_1 - 2M = 36.6140 - 17.2332$$
$$= 19.3808 \text{ microhenrys.}$$

as found above by formula (100).

Taking account of the ends neglected above, we should find that $2L_1$ for the two ends by (95) is 181.9 cm and $2M$ by (98) is practically zero. Hence the self-inductance of the circuit is, including the ends,

$$L = 19.5627 \text{ microhenrys.}$$

EXAMPLE 73. FORMULA (102) FOR THE MUTUAL INDUCTANCE OF ADJACENT CONDUCTORS IN THE SAME STRAIGHT LINE

When the two conductors are of equal length, $l = m$, and (102) becomes

$$M = 2\,l \log_e 2 = 2\,l \times 0.69315 \text{ cm.}$$

If $l = 1000$ cm, $M = 1386.3$ cm.

If $m = 1000\ l$, (81) gives

$$\begin{aligned} M &= l \log_e 1001 + 1000\ l \log 1.001 \\ &= l \log_e 1001 + l \text{ approximately.} \end{aligned}$$

If $l = 1$ cm, we have

$$\begin{aligned} M &= \log_e 1001 + 1000 \log_e 1.001 \\ &= 6.909 + 0.999 = 7.908 \text{ cm.} \end{aligned}$$

The self-inductance of the short wire AB, supposed 1 cm long and of 1 mm radius, is

$$L = 2\left(\log_e \frac{2}{0.1} - .75\right) = 2\ (2.9957 - .75) = 4.4915 \text{ cm,}$$

which is a little more than one-half of the mutual inductance of AB and BC, BC being one thousand times the length of AB.

In closed circuits, all the magnetic lines due to a circuit are effective in producing self-inductance, and hence the self-inductance is always greater than the mutual inductance of that circuit with any other, assuming one turn in each. But with open circuits, as in this case, we may have a mutual inductance between two single conductors greater than the self-inductance of one of them.

EXAMPLE 74. FORMULA (104) FOR THE SELF-INDUCTANCE OF A RECTANGULAR BAR

In formula (104), substituting $l = 1000$, and $\alpha + \beta = 2$ for a square bar 1000 cm long and 1 square cm section, we have, neglecting the small last term,

$$\begin{aligned} L &= 2000\left[\log_e \frac{2000}{2} + \frac{1}{2}\right] \\ &= 2000\ (6.908 + 0.5) = 14816 \text{ cm} \\ &= 14.816 \text{ microhenrys.} \end{aligned}$$

This would also be the self-inductance for any section having $\alpha + \beta = 2$ cm.

EXAMPLE 75. FORMULAS (105) AND (106) FOR THE SELF-INDUCTANCE OF A SQUARE MADE UP OF A ROUND WIRE

If the side of the square is 1 meter, $a = 100$ cm, $\rho = 0.1$ cm, we have from (105)

$$\begin{aligned} L &= 800\ (\log_e 1000 - 0.524) \\ &= 5107 \text{ cm} = 5.107 \text{ microhenrys.} \end{aligned}$$

If $\rho = .05$ cm,

$$L = 5662 \text{ cm} = 5.662 \text{ microhenrys.}$$

That is, the self-inductance of such a rectangle of round wire is about 11 per cent greater for a wire 1 mm in diameter than for one 2 mm in diameter.

If l/ρ is constant, L is proportional to l, that is, if the thickness of the wire is proportional to the length of the wire in the square, the self-inductance of the square is proportional to its linear dimensions.

EXAMPLE 76. FORMULA (107) FOR THE SELF-INDUCTANCE OF A RECTANGLE OF ROUND WIRE

Suppose a rectangle 2 meters long and 1 meter broad.

Substituting $a = 200$ cm, $b = 100$, $\rho = 0.1$, in (107) we have

$$L = 8017.1 \text{ cm} = 8.017 \text{ microhenrys.}$$

We can obtain the same result from the values of self and mutual inductances calculated in examples 70 and 71. That is, the resultant self-inductance of the rectangle is the sum of the self-inductances of the four sides, minus twice the mutual inductances of the two pairs of opposite sides. Thus

$$L = (L_1 + L_3) + (L_2 + L_4) - 2M_{13} - 2M_{24}$$

$$\begin{array}{lrrr}
\text{By example 70,} & L_1 + L_3 = & 6035.24 & \\
& L_2 + L_4 = & \underline{2740.36} & 8775.60 \\
\text{By example 71,} & 2M_{13} = & 660.48 & \\
& 2M_{24} = & \underline{98.04} & \underline{758.52} \\
& & \therefore L = & 8017.08 \text{ cm} \\
& & = & 8.0171 \text{ microhenrys.}
\end{array}$$

The agreement of this result with that obtained from formula (107) serves as a check on the latter formula, and also illustrates how the values of the self and mutual inductances of open circuits may be combined to give the self-inductance of a closed circuit.

EXAMPLE 77. FORMULAS (108), (109), AND (110) FOR THE SELF-INDUCTANCE OF A RECTANGLE OR SQUARE MADE UP OF A BAR OF RECTANGULAR SECTION

Let $a=200 \qquad b=100 \qquad \alpha=\beta=1.0$ cm.

Substituting these values in (108) we obtain

$$\begin{aligned} L &= 4(2971.05-1209.76-577.95-150+447.21+0.99) \\ &= 5926.16 \text{ cm.} \end{aligned}$$

For a square 10 meters on a side, made of square bar 1 sq cm cross section we have $a=1000$, $\alpha=1$; substituting in (110)

$$\begin{aligned} L &= 8000(6.908+.033) \\ &= 8000 \times 6.941 \text{ cm} = 55.53 \text{ microhenrys.} \end{aligned}$$

For a circular section, diameter 1 cm, $\rho=0.5$; substituting in (105)

$$\begin{aligned} L &= 8000\left(\log_e 2000+\frac{1}{2000}-0.524\right) \\ &= 8000 \times 7.076 \text{ cm} = 56.61 \text{ microhenrys,} \end{aligned}$$

a little more than for a square section, as would be expected.

EXAMPLE 78. FORMULA (112) FOR THE MUTUAL INDUCTANCE OF PARALLEL SQUARES

Suppose two parallel squares each 1 meter on a side, 10 cm distant from one another.

$a=100$, $d=10$. Substituting in (112),

$$\begin{aligned} M &= 8\left[100 \log_e\left(\frac{1+\sqrt{1.01}}{1+\sqrt{2.01}}\cdot\frac{\sqrt{101}}{1}\right)+\sqrt{20100}-2\sqrt{10100}+10\right] \\ &= 800\left[\log_e\left(\frac{10.1+\sqrt{101}}{1+\sqrt{2.01}}\right)+\sqrt{2.01}-2\sqrt{1.01}+0.1\right] \\ &= 1142.5 \text{ cm} = 1.1425 \text{ microhenrys.} \end{aligned}$$

EXAMPLE 79. FORMULAS (113), (114), AND (115) FOR THE SELF AND MUTUAL INDUCTANCE OF THIN STRAIGHT STRIPS OR TAPES

Let the tape of thin copper be 10 meters long and 1 cm wide.

Substituting $l=1000$ and $b=1$ in (113) we have

$$L=2000\left(\log_e 2000+\frac{1}{2}\right)$$
$$=2000\times 8.1009=16202 \text{ cm}$$
$$=16.202 \text{ microhenrys,}$$

as the self-inductance when the conducting strip is very thin. If the tape is 2 mm thick we may allow for the effect of the thickness by using (114) and we find

$$L=2000\times 7.9009 \text{ cm}=15.802 \text{ microhenrys,}$$

which differs slightly from the preceding value.

Two such tapes edge to edge in one plane will have a mutual inductance by (115) of

$$M=2000\ (\log_e 2000-0.8863)$$
$$=2000\times 6.7146 \text{ cm}$$
$$=13.429 \text{ microhenrys.}$$

EXAMPLE 80. FORMULA (117) FOR THE SELF-INDUCTANCE OF A RETURN CIRCUIT OF TWO PARALLEL SHEETS; NONINDUCTIVE SHUNTS

Suppose the dimensions of a thin manganin sheet which has been doubled on itself be as follows:

$$l=30 \text{ cm} \qquad b=10 \text{ cm} \qquad d=1 \text{ cm.}$$

By (132) $\log R_2=1.0787$

$$\log R_1=\log_e 10-\frac{3}{2}=0.8026$$
$$L=4l\ (\log R_2-\log R_1)$$
$$=120\times 0.2761$$
$$=33.13 \text{ cm}$$
$$=.0331 \text{ microhenrys.}$$

EXAMPLE 81. FORMULA (122), 3 CONDUCTORS IN MULTIPLE

Suppose three cylindrical conductors, each 10 meters long and 4 mm diameter, the distance apart of their centers being 1 cm. Substitute in (122) as follows:

$$l=1000 \text{ cm} \qquad \rho=2 \text{ mm} \qquad d=1 \text{ cm.}$$

Then $(r_g a^2)^{\frac{1}{3}} = 0.538$ cm

and
$$L = 2000\left(\log_e \frac{2000}{0.538} - 1\right)$$
$$= 2000 \times 7.221 \text{ cm} = 14.442 \text{ microhenrys.}$$

If the whole current flowed through a single one of the three conductors the self-inductance would be

$$L = 2000\left(\log_e \frac{2000}{0.2} - \frac{3}{4}\right) = 17.92 \text{ microhenrys,}$$

or about 25 per cent more than when divided among the three.

9. FORMULAS FOR GEOMETRICAL AND ARITHMETICAL MEAN DISTANCES

GEOMETRICAL MEAN DISTANCES

Maxwell showed how to calculate mutual and self-inductances in several important cases by means of what he called the geometrical mean distance, either of one conductor from another or of a conductor from itself. On account of the importance of this method we give below some of the most useful of these formulas. The geometrical mean distance of a point from a line is the n^{th} *root of the product of the n distances from the point P to the various points in the line, n being increased to infinity in determining the value of R.* Or, the logarithm of R is the *mean value of log d for all the infinite values of the distance d.* Similarly, the geometrical mean distance of a line from itself is the n^{th} *root of the product of the n distances between all the various pairs of points in the line, n being infinity.*[109]

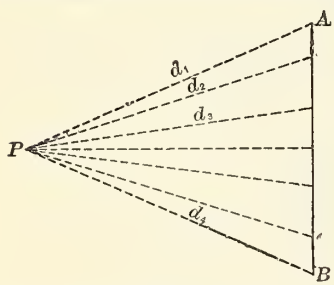

Fig. 50

Similar definitions apply to the g. m. d. of one area from another, or of an area from itself.

[109] Rosa, this Bulletin, 4, p. 325; 1907.

The geometrical mean distance R of a *line* of length a from itself is given by

$$\log R = \log a - \frac{3}{2}$$

$$R = ae^{-\frac{3}{2}} \qquad [123]$$

$$\text{or } R = 0.22313a$$

The g. m. d. of a *rectangular area* of sides a and b from itself is given by

$$\log R = \log\sqrt{a^2+b^2} - \frac{1}{6}\frac{a^2}{b^2}\log\sqrt{1+\frac{b^2}{a^2}} - \frac{1}{6}\frac{b^2}{a^2}\log\sqrt{1+\frac{a^2}{b^2}}$$

$$+\frac{2}{3}\frac{a}{b}\tan^{-1}\frac{b}{a} + \frac{2}{3}\frac{b}{a}\tan^{-1}\frac{a}{b} - \frac{25}{12} \qquad [124]$$

When the area is a *square*, and hence $a = b$,

$$\log R = \log a + \frac{1}{3}\log 2 + \frac{\pi}{3} - \frac{25}{12}$$

$$\therefore\ R = 0.44705\ a \qquad [125]$$

For a *circular area* of radius a,

$$\log R = \log a - \frac{1}{4}$$

$$R = ae^{-\frac{1}{4}} \qquad [126]$$

$$R = 0.7788\ a$$

For an *ellipse* of semi-axes a and b,

$$\log R = \log\frac{a+b}{2} - \frac{1}{4} \qquad [127]$$

An approximate expression for the g. m. d. of a *rectangular area* of length a and breadth b is

$$R = 0.2235(a+b) \qquad [128]$$

which is nearly true for all values of a and b; that is, the geometrical mean distance of the rectangular area from itself is approximately proportional to the perimeter of the rectangle. The following table gives the ratio $R/(a+b)$ for a series of rectangles of different proportions, from a square to a ratio of 20 to 1 between length and breadth, and finally when the breadth is infinitesimal in comparison with the length. By interpolating for any other case between the

values given in the table one can obtain a quite accurate value without the trouble of calculating it by formula (124).

Geometrical Mean Distances of Rectangles of Different Proportions

[a and b are the Length and Breadth of the Rectangles. R is the Geometrical Mean Distance of its Area]

Ratio	R	$\frac{R}{a+b}$
1 :1	0.44705a	0.22353
1.25:1	0.40235a	0.22353
1.5 :1	0.37258a	0.22355
2 :1	0.33540a	0.22360
4 :1	0.27961a	0.22369
10 :1	0.24596a	0.22360
20 :1	0.23463a	0.22346
1 :0	0.22313a	0.22313

The g. m. d. of an *annular area* of radii a_1 and a_2 from itself is given by

$$\log R = \log a_1 - \frac{a_2^4}{(a_1^2 - a_2^2)^2} \log \frac{a_1}{a_2} + \frac{1}{4} \frac{3a_2^2 - a_1^2}{a_1^2 - a_2^2} \qquad [129]$$

The g. m. d. of a *line* of length a from a second line of the same length, distant in the same straight line na, center to center, Fig. 51, is given by the following formula:

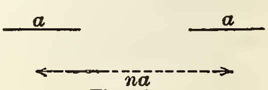

Fig. 51

$$\log R_n = \frac{(n+1)^2}{2} \log (n+1)a - n^2 \log na + \frac{(n-1)^2}{2} \log (n-1)a - \frac{3}{2} \qquad [130]$$

This formula is equivalent to the following, which is more convenient for calculation for all values of n greater than one.[110]

$$\log R_n = \log n - \left[\frac{1}{12n^2} + \frac{1}{60n^4} + \frac{1}{168n^6} + \frac{1}{360n^8} + \frac{1}{660n^{10}} + \cdots \right] \qquad [131]$$

This formula is very convergent, and only two or three terms are generally required.

[110] Rosa, this Bulletin, 2, p. 168: 1906.

The following values of the geometrical mean distances (calling a unity) were calculated from the above formulas, all after the second being obtained by (131):

$R_0 = 0.22313$	$R_5 = 4.98323$
$R_1 = 0.89252$	$R_6 = 5.98610$
$R_2 = 1.95653$	$R_7 = 6.98806$
$R_3 = 2.97171$	$R_8 = 7.98957$
$R_4 = 3.97890$	$R_9 = 8.99076$

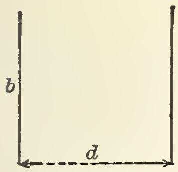

Fig. 52

If the lines are parallel and at distance d, Fig. 52, the g. m. d. is given by

$$\log R = \frac{d^2}{b^2}\log d + \frac{1}{2}\left(1 - \frac{d^2}{b^2}\right)\log (b^2 + d^2) + 2\frac{d}{b}\tan^{-1}\frac{b}{d} - \frac{3}{2} \qquad [132]$$

If $d = b$,

$$\log R = \log b + \frac{\pi}{2} - \frac{3}{2} \qquad [133]$$

The g. m. d. from a point O_2, Fig. 53, outside a circle to the circumference of the circle, or to the entire area of the circle *is the distance d from O_2 to the center of the circle.*

(1) The g. m. d. from the center O_1 to the circumference is of course the radius a. (2) The g. m. d. of any point (as O_3) within the circle from the circumference is also a. (3) The g. m. d. of any point on the circumference (as O_4) from all other points of the circumference is also a. (4) Therefore the g. m. d. of a circular line of radius a from itself is a; that is,

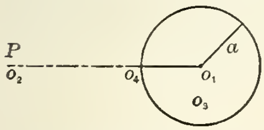

Fig. 53

$$R = a \qquad [134]$$

for each of the four cases named above.

The g. m. d. of a point outside a circular ring, Fig. 54, from the ring *is the distance d to the center of the ring.* The g. m. d. of any point O_1, O_3, etc., within the ring is given by

$$\log R = \frac{a_1^2 \log a_1 - a_2^2 \log a_2}{a_1^2 - a_2^2} - \frac{1}{2} \qquad [135]$$

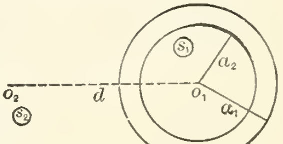

Fig. 54

The same expression gives the g. m. d. of any figure, as S_1, within the ring from the ring. The g. m. d. of an external figure,

as S_2, from the annular ring is equal to the g. m. d. of the center O_1 from the figure S_2.

The g. m. d. from one circular area to another is the distance between their centers; that is,

$$R = d \qquad [136]$$

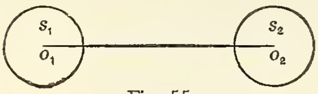

Fig. 55

for the area S_1 with respect to S_2 as it is for the point O_1 with respect to S_2.

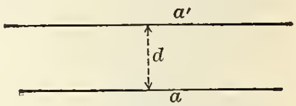

Fig. 56

The g. m. d. of a line of length a from a second parallel line of length a' located symmetrically (Fig. 56) is given by Gray[111], equation (114). The g. m. d. of a line from a parallel and symmetrically situated rectangle is given by Gray's equation (112). The g. m. d. of two unequal rectangles from one another is given by Gray's equation (113).[112]

The g. m. d. of two adjacent rectangles and of two obliquely situated rectangles are given by Rosa,[113] equations (8*a*) and (17). As these expressions are somewhat lengthy and not often required they are not repeated here. The values of the g. m. d. for two equal squares in various relative positions to one another have been accurately calculated[114] by these formulas, and the results used in the determination[115] of the correction term E of formula (93).

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[111] Absolute Measurements, Vol. II, Part I.

There are a number of misprints in equations 104, 109, 111, and 113 of Gray. The sign of the first term of equation 104 should be +. The signs before p^2 in the coefficients of the log in the first four terms of equation 109 should be all minus; thus $\frac{1}{4}(\beta^2-p^2), -\frac{1}{4}(\alpha^2-p^2), -\frac{1}{4}[(\alpha-\beta)^2-p^2], +\frac{1}{4}[(a-\alpha)^2-p^2]$. Similarly in equation 111 the coefficients of the first two terms should be $\frac{1}{2}(\beta^2-p^2)$ and $-\frac{1}{2}(\alpha^2-p^2)$. In equation 113 the coefficient of β^4 in each of the first four terms should be $\frac{1}{6}$ instead of $\frac{1}{2}$ and the first term should have log $[(p+b+b')^2+\beta^2]$ instead of log $[(p+b+b')^2-\beta^2]$.

[112] Also by Rosa, equation (8) this Bulletin, **3**, p. 6; 1907.

[113] This Bulletin, **3**, pp. 7 and 12, 1907.

[114] This Bulletin, **3**, pp. 9–19; 1907.

[115] This Bulletin, **3**, p. 37; 1907.

ARITHMETICAL MEAN DISTANCES

In the determination of self and mutual inductances by the method of geometrical mean distances it has been shown[116] that more accurate formulas can be obtained by the use of certain arithmetical mean distances and arithmetical mean square distances taken in connection with geometrical mean distances.

The arithmetical mean distance of a point from a line is the arithmetical mean of the n distances of the point from the various points of the line, n being infinite. Similarly, the arithmetical mean distance of a line from itself is the *arithmetical mean of the distances of the n pairs of points in the line from one another, n being infinite.*

The a. m. d. of a line of length b from itself is[117]

$$S_2=\frac{b}{3} \qquad [137]$$

that is, while the g. m. d. of a line from itself is 0.22313 times its length, the a. m. d. is one-third the length.

The arithmetical mean square distance of a line from itself is of course larger than the square of the a. m. d. Putting S_2^2 for the arithmetical mean square distance (a. m. s. d.).

$$S_2^2=\frac{b^2}{6} \text{ or } \sqrt{S_2^2}=\frac{b}{\sqrt{6}} \qquad [138]$$

The arithmetical mean distance of a point in the circumference of a circle from the circle is the same as the a. m. d. of the circle from itself; that is, for a circle of radius a,

$$S_1=S_2=\frac{4}{\pi}a \qquad [139]$$

The arithmetical mean square distance is

$$S_2^2=2a^2 \text{ and } \sqrt{S_2^2}=a\sqrt{2} \qquad [140]$$

(The g. m. d. for this case is $R=a$, equation (134).)

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[116] Rosa, this Bulletin, 4, pp. 326–32; 1907.
[117] Rosa, this Bulletin, 4, p. 326; 1907.

The arithmetical mean distance of an external point P from the circumference of a circle, Fig. 57, is

$$S_1 = \sqrt{d^2 + a^2} \qquad [141]$$

which is the distance PA.

The arithmetical mean distance from P to the entire area of the circle is

$$S_1 = \sqrt{d^2 + \frac{a^2}{2}} \qquad [142]$$

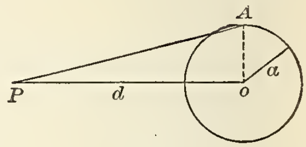

Fig. 57

(The g. m. d. for each of these cases is $R = d$, equation (136).)

For the proof of these and other expressions for the arithmetical mean distances and applications of their use see the article referred to above.

10. HIGH-FREQUENCY FORMULAS

Excepting in a very few specified cases, the formulas of the preceding sections apply only to conductors carrying direct current or alternating currents of frequencies so low that the error, due to the assumption that the current is uniformly distributed over the cross section of the wire, is negligible.

In the case of standards of mutual inductance the inductance may be regarded as sensibly independent of the frequency, unless the two coils are very close together, and even then the capacity between the coils will be a more potent source of error than the departure of the current from a uniform distribution over the cross section of the wire.

The self-inductance of a coil or conductor, on the other hand, depends appreciably on the field in the cross section of the conductor, and any deviation of the distribution of the current in the wire from uniformity gives rise to a *decrease* in the inductance. The amount of this change depends on the frequency of the current and the radius of the cross section of the conductor, as well as on the conductivity and permeability of the material of which it is composed.

This *decrease* of the inductance is accompanied by an *increase* in the resistance of the conductor. Whereas, however, the inductance

with increasing frequency approaches a limiting value, the resistance increases indefinitely as the frequency approaches an infinite value. The change of resistance is always relatively much larger than the change in inductance.

The eddy current effects just described are, for the most part, negligible at low frequencies, except in the case of heavy conductors and in coils wound with stout wire in several layers. In the latter case, however, the diminution of the inductance, due to the irregular distribution of the current, is marked, to a greater or less degree, by the effect of the capacity between the windings of the coil, which gives rise to an *increase* of the inductance with the frequency. For the same reason the resistance is increased more than it would be by the eddy currents alone.

Unfortunately, the rigorous or approximate solution of the problem at high frequencies for the various cases for which the inductance with steady currents may be calculated is in many instances very difficult, if not impossible. Some of the simpler cases, however, because of their great importance, have received much attention, with the result that the changes of inductance and resistance may be calculated with a good degree of precision.

STRAIGHT CYLINDRICAL WIRES

This is the most important case of all, since the solution is rigorous, and the results may be applied to the construction of practical, absolute standards for high-frequency work. The problem has been treated successively by Maxwell,[118] Heaviside,[119] Rayleigh,[120] and Kelvin.[121]

Putting l = length of conductor
ρ = radius of conductor
σ = specific resistance of its material
μ = permeability
f = frequency, $p = 2\pi f$
R' = resistance with current of frequency f
L' = inductance " " " " f

[118] Elect. and Mag., II, § 690.
[119] Elect. Papers, II, p. 64.
[120] Phil. Mag., 21, p. 381; 1886.
[121] Math. and Phys. Papers, III, p. 491; 1889.

R = resistance with direct current
L = inductance " " "

$$x = 2\rho\sqrt{\frac{\pi p \mu}{\sigma}}$$

Thus

$$\frac{R'}{R} = \frac{x}{2}\frac{W}{Y} \qquad [143]$$

$$L' = 2l\left[\log\frac{2l}{\rho} - 1 + \frac{\mu}{4}\left(\frac{4}{x}\frac{Z}{Y}\right)\right] \qquad [144]$$

where

$$W = \text{ber}\, x \,\text{bei}'\, x - \text{bei}\, x \,\text{ber}'\, x$$
$$Y = (\text{ber}'\, x)^2 + (\text{bei}'\, x)^2$$
$$Z = \text{ber}\, x \,\text{ber}'\, x + \text{bei}\, x \,\text{bei}'\, x \qquad [144a]$$

Since from (96)

$$L = 2l\left[\log\frac{2l}{\rho} - 1 + \frac{\mu}{4}\right]$$

we find

$$\Delta L = L' - L = -2l\frac{\mu}{4}\left[1 - \frac{4}{x}\frac{Z}{Y}\right] \qquad [145]$$

$$\frac{\Delta L}{L} = -\frac{\frac{\mu}{4}\left(1 - \frac{4}{x}\frac{Z}{Y}\right)}{\log\frac{2l}{\rho} - 1 + \frac{\mu}{4}} \qquad [146]$$

For nonmagnetic material the equation (146) takes the form

$$\frac{\Delta L}{L} = -\frac{\left(1 - \frac{4}{x}\frac{Z}{Y}\right)}{4\log\frac{2l}{\rho} - 3} \qquad [147]$$

In these expressions, ber x and bei x are functions introduced by Lord Kelvin, being respectively the real and imaginary parts of the ordinary Bessel function of order zero, J_0, having for its argument $xi\sqrt{i}$, where x is a real quantity, and $i = \sqrt{-1}$. These functions are given by the series

$$\left.\begin{aligned} \operatorname{ber} x &= 1 - \frac{x^4}{2^2 4^2} + \frac{x^8}{2^2 4^2 6^2 8^2} - \cdots\cdots \\ \operatorname{bei} x &= \frac{x^2}{2^2} - \frac{x^6}{2^2 4^2 6^2} + \frac{x^{10}}{2^2 4^2 6^2 8^2 10^2} - \cdots\cdots \end{aligned}\right\} \qquad [148]$$

and $\operatorname{ber}'x$ and $\operatorname{bei}'x$ are their differential coefficients with respect to x.

These series are very convergent, but the calculation, naturally, becomes laborious for large values of x. To lighten the labor of calculation Russell[122] and Savidge[123] have developed asymptotic expressions for ber x, bei x, ber $'x$, bei $'x$ and the auxiliary quantities W, Y, and Z, which give their numerical values with an accuracy of about one part in ten thousand for values of x greater than about 6, but whose accuracy increases rapidly as x becomes larger.

Savidge[123] has, in addition, calculated extensive tables of the above functions and the allied ker and kei functions to four places of decimals, and for values of the argument ranging between 1 and 30 in steps of one unit. These tables will be found very useful in the solution of a variety of problems. For calculation with the formulas (143) to (147), however, it seemed desirable to construct tables in which the argument advances by smaller steps than in the tables of Savidge. For this purpose ber x, bei x, ber $'x$ and bei $'x$ were calculated directly from their series, for arguments from 0.1 to 5.0, in steps of 0.1, and from 5 to 7 in steps of 0.2. From these were obtained directly by (144*a*) the values of W, Y, Z. For the larger values of x, the quantities W, Y, Z were calculated by asymptotic formulas, and checked at a few points by the direct series. Thus the interval from 5 to 10 was covered in steps of 0.2, the interval 10 to 15 in steps of 0.5, and from 15 to 50 in steps of one unit, the aim being to keep the differences of the same order of magnitude. It will, probably, seldom be necessary to make calculations for values of x greater than about 50. If such calculations are occasionally required, they may be made with little trouble by the asymptotic formulas given below.

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[122] Phil. Mag., **17**, p. 524; 1909.
[123] Phil. Mag., **19**, p. 49; 1910.

Since making the above calculations, we have determined the expressions for the general terms in Russell's equations (8), (9), and (10), (loc. cit., p. 529), thus materially increasing their range of applicability.

These equations, thus extended, are

$$\left.\begin{aligned}
W&=\frac{x}{2}\Big\{1+\frac{6}{(|\underline{3})^2}\left(\frac{x}{2}\right)^4+\frac{30}{(|\underline{5})^2}\left(\frac{x}{2}\right)^8+\frac{140}{(|\underline{7})^2}\left(\frac{x}{2}\right)^{12}+\cdots\\
&\qquad+\frac{1\cdot2\cdot3\cdots\cdot(2n+1)}{1^2\,2^2\,3^2\cdots n^2(|\underline{2n+1})^2}\left(\frac{x}{2}\right)^{4n}+\cdots\Big\}\\
Y&=\frac{x^2}{4}\Big\{1+\frac{3}{(|\underline{3})^2}\left(\frac{x}{2}\right)^4+\frac{10}{(|\underline{5})^2}\left(\frac{x}{2}\right)^8+\frac{35}{(|\underline{7})^2}\left(\frac{x}{2}\right)^{12}+\cdots\\
&\qquad+\frac{1}{(n+1)}\,\frac{1\cdot2\cdot3\cdot\cdots\cdot(2n+1)}{1^2\,2^2\,3^2\cdots n^2\,(|\underline{2n+1})^2}\left(\frac{x}{2}\right)^{4n}+\cdots\Big\}\\
Z&=\frac{x^3}{16}\Big\{1+\frac{3}{2(|\underline{3})^2}\left(\frac{x}{2}\right)^4+\frac{10}{3(|\underline{5})^2}\left(\frac{x}{2}\right)^8+\frac{35}{4(|\underline{7})^2}\left(\frac{x}{2}\right)^{12}+\cdots\\
&\qquad+\frac{1}{(n+1)^2}\,\frac{1\cdot2\cdot3\cdot\cdots\cdot(2n+1)}{1^2\,2^2\,3^2\cdots\cdot n^2(|\underline{2n+1})^2}\left(\frac{x}{2}\right)^{4n}+\cdots\Big\}
\end{aligned}\right\}\quad[149]$$

It would have been simpler to have calculated the values of W, Y, and Z by these formulas than by the more indirect process actually used. The formulas (149) have, however, shown themselves of great service in checking the results. For completeness the asymptotic formulas of Savidge used have also been appended. They give results to one in one hundred thousand for $x \geqq 10$ and may be used with an error of not greater than one in ten thousand down to $x=6$.

$$\left.\begin{aligned}
W&=\frac{e^{x\sqrt{2}}}{2\pi x}\left[\frac{1}{\sqrt{2}}+\frac{1}{8x}+\frac{9}{(8x)^2\sqrt{2}}+\frac{39}{(8x)^3}+\frac{75}{2(8x)^4\sqrt{2}}-\cdots\right]\\
Y&=\frac{e^{x\sqrt{2}}}{2\pi x}\left[1-\frac{6}{8x\sqrt{2}}+\frac{9}{(8x)^2}+\frac{150}{(8x)^3\sqrt{2}}+\frac{2475}{2(8x)^4}+\cdots\right]\\
Z&=\frac{e^{x\sqrt{2}}}{2\pi x}\left[\frac{1}{\sqrt{2}}-\frac{3}{8x}-\frac{15}{(8x)^2\sqrt{2}}-\frac{45}{(8x)^3}+\frac{315}{2(8x)^4\sqrt{2}}+\cdots\right]
\end{aligned}\right\}\quad[150]$$

The results are given in Table XXII which gives the values, to one in one hundred thousand, of not only the quantities $\frac{x}{2}\frac{W}{Y}$ and

$\frac{4}{x}\frac{Z}{Y}$ required in the preceding formulas, but of $\frac{W}{Y}$ and $\frac{Z}{Y}$ also. These will be found useful in allied problems, and it may seem preferable in some cases to interpolate the values of these latter quantities to obtain the former. For example, with $x > 2.5$ the first differences, and in some places the second differences also, are smaller with $\frac{W}{Y}$ than with $\frac{x}{2}\frac{W}{Y}$. The accuracy of the Table XXII may be regarded as greater than will usually be required, and should suffice for the most precise work.

In addition to the general formulas of Kelvin (143) and (144), Rayleigh[124] has given expansions holding for small values of the argument x. These equations, which were extended to another term by Heaviside, are, expressed in the present nomenclature,

$$\left.\begin{aligned} \frac{x}{2}\frac{W}{Y} &= 1 + \frac{1}{12}\left(\frac{x}{2}\right)^4 - \frac{1}{180}\left(\frac{x}{2}\right)^8 + \frac{11}{12 \cdot 28 \cdot 30}\left(\frac{x}{2}\right)^{12} - \cdots \\ \frac{4}{x}\frac{Z}{Y} &= 1 - \frac{1}{24}\left(\frac{x}{2}\right)^4 + \frac{13}{4320}\left(\frac{x}{2}\right)^8 - \frac{647}{12^2 \cdot 360 \cdot 56}\left(\frac{x}{2}\right)^{12} + \cdots \end{aligned}\right\} \quad [151]$$

Their applicability is limited to the range of values of x less than about 2, and it will be more convenient to use Table XXII.

For very high frequencies Rayleigh gave also the limiting formulas

$$R' = \sqrt{\frac{1}{2}pl\mu R}$$

$$L' = 2l\left[\log\frac{2l}{\rho} - 1 + \frac{1}{2}\sqrt{\frac{\mu R}{2pl}}\right]$$

In some instances these formulas have been used, as though they were exact, over a considerable range of frequencies, without any statement being made as to the magnitude of the error involved. Expressing these formulas in the present nomenclature, we obtain the following formulas for infinite frequencies:

[124] Phil. Mag., **21**, p. 387; 1886.

$$\left(\frac{R'}{R}\right)_{x=\infty}=\frac{x}{2\sqrt{2}} \qquad [152]$$

$$(L')_{x=\infty}=2l\left[\log\frac{2l}{\rho}-1+\frac{\mu}{4}\left(\frac{4}{x\sqrt{2}}\right)\right]_{x=\infty}$$

$$=2l\left[\log\frac{2l}{\rho}-1\right] \qquad [153]$$

These are seen to be in agreement with equations (143) and (144), if we remember that the limiting values of $\frac{W}{Y}$ and $\frac{Z}{Y}$ as the argument x is indefinitely increased are both $\frac{1}{\sqrt{2}}$. (See formulas (150).) From (150) we find that only for values of x greater than about 900 is the error from using (152) as small as one-tenth per cent. For $x=70$ the error is about 1 per cent, and in many practical cases it is still larger.

The limiting value of the change of inductance is found from (147) to be

$$\left(\frac{\Delta L}{L}\right)_{x=\infty}=-\frac{1}{4\log\frac{2l}{\rho}-3} \qquad [154]$$

$$(\Delta L)_{x=\infty}=-\frac{l}{2} \qquad [155]$$

The error from using (153) is only about one part in ten thousand for $x=60$. The error, however, arising from the neglect of the term $\frac{4}{x}\frac{Z}{Y}$ in (147) is more than 5 per cent.

From (154) we obtain the curious result that the limiting value of the fractional change of inductance, as the frequency is indefinitely increased, depends only on the ratio of the length of the wire to the cross section. Table XXIII gives an idea of the way the limiting value falls off as this ratio is increased.

The preceding formulas show that the change of resistance and inductance are functions of the quantity

$$x=2\rho\sqrt{\frac{\pi\mu p}{\sigma}}=2\pi\rho\sqrt{2\mu Kf} \qquad [156]$$

where K is the conductivity.

Taking the specific resistance of annealed copper at 20° as 1.721 microhms or 1721 in absolute electromagnetic units,

$$K_0 = 5.811 \times 10^{-4}$$

and (156) takes the simple form

$$x = 0.2142 \rho \sqrt{f}$$

To aid in making approximate calculations, and for purposes of orientation, the auxiliary Table XXIV has been calculated, giving the value $x = x_0$ for copper wire of the above conductivity and of a cross section of 1 mm radius at various frequencies. For the higher frequencies, the corresponding wave length λ in meters has been included as likely to be of service in calculations for wireless-telegraph circuits. The range of this table may be considerably extended by remembering that x varies with $\sqrt{f}$ or $\sqrt{\frac{1}{\lambda}}$. Thus the value of x_0 for 7 500 cycles is found directly from the tabulated value for 750 000 cycles by shifting the decimal point. Similarly, the value for $\lambda = 150$ meters is obtained from the tabulated value for 15 000 meters. It is for this reason that the larger values of λ have been tabulated.

To calculate x for a copper wire of radius r mm, we have $x = x_0 r$, and if the conductivity have any value K, the further factor $\sqrt{\frac{K}{K_0}}$ must be applied. Finally, if the wire is, in addition, of magnetic material of permeability μ, an additional factor $\sqrt{\mu}$ is necessary to obtain the required value of x.

CONCENTRIC MAIN

The simple case of a cylindrical, straight wire may be regarded as a special case of the more general problem of a concentric main; that is, of a solid or tubular inner conductor surrounded by a coaxial tubular outer conductor. This case has been very completely treated by Russell,[125] but as the formulas are not simple they are not given here.

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[125] Phil. Mag., **17**, p. 524; 1909.

TWO PARALLEL WIRES

Unless the two wires are so near together, relatively to their radius of cross section, that their mutual inductance is appreciably affected by changes in the distribution of the current within the wires, each wire may be treated by the formulas given for a straight, cylindrical wire.

Supposing, therefore, that the wires are alike in every respect

$$L' = 4l\left[\log\frac{d}{\rho} + \frac{\mu}{4}\left(\frac{4}{x}\frac{Z}{Y}\right)\right] \qquad [157]$$

and from (101) we find for wires of nonmagnetic material

$$\frac{\Delta L}{L} = -\frac{\left(1-\frac{4}{x}\frac{Z}{Y}\right)}{4\log\frac{d}{\rho}+1} \qquad [158]$$

$$\left(\frac{\Delta L}{L}\right)_{x=\infty} = -\frac{1}{4\log\frac{d}{\rho}+1} \qquad [159]$$

$$(\Delta L)_{x=\infty} = l \qquad [159a]$$

and

$$\frac{R'}{R} = \frac{x}{2}\frac{W}{Y} \qquad [160]$$

the values of $\frac{Z}{Y}$ and $\frac{W}{Y}$ being taken from Table XXII.

Nicholson[126] has recently given a solution of the problem, when the two wires are so close together that their mutual inductance suffers a sensible change with the frequency. To obtain an idea of the magnitude of this effect, in a practical case, the results by Nicholson's formulas were compared with those of (158) and (160). With $d = 1$ cm and $\rho = 0.1$ cm, and with a frequency of 10^6, (158) gives $\frac{\Delta L}{L} = -8.5$ per cent, the effect of Nicholson's correction being to give a value of $\frac{\Delta L}{L}$ numerically only nine parts in ten thousand smaller.

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[126] Phil. Mag., 18, p. 417; 1909.

Similarly for the resistance, (160) gives $\frac{R'}{R}=7.56$, while Nicholson's formula reduces this to 7.55. Since this example relates to a rather unfavorable case, for a standard whose inductance is to be calculated from the dimensions, these corrections for mutual effect may, in general, be regarded as negligible, and the formulas (158), (159), and (160) may be regarded as sufficiently accurate with the precision usually attainable in the measurement of the dimensions.

It is to be noticed that the maximum possible relative change of inductance, with the frequency, is greater with two parallel wires than with either alone, because this change with the parallel wires depends on the sum of their self-inductances which is greater than the resulting self-inductance of the combination (see p. 151). Table XXIII gives an idea of the values attained by $\left(\frac{\Delta L}{L}\right)_{x=\infty}$ in the case of the two parallel wires. This maximum change of inductance depends only on the ratio of their distance apart to the radius of cross section of the wire.

Evidently, other cases of linear conductors of circular cross section, may likewise be made to depend on the solution for straight wires.

CIRCULAR RING OF CIRCULAR SECTION

The inductance of a circular ring, in which the current is confined wholly to the circumference of the cross section was given in formula (65). Combining this with (63) we find that on the assumption that (65) represents the actual distribution of the current at infinite frequency,

$$(\Delta L)_{x=\infty}=-\pi a\left[1-\frac{1}{2}\frac{\rho^2}{a^2}\left(\log\frac{8a}{\rho}-\frac{1}{3}\right)\right] \qquad [161]$$

The absolute value of the change of inductance at infinite frequency is, in the case of a straight wire, (see 145)

$$(\Delta L)_{x=\infty}=-\frac{l}{2}$$

which shows that, if the wire of the ring were stretched out straight, the value given in (161) would become

$$(\Delta L)_{x=\infty}=-\frac{1}{2}2\pi a=-\pi a \qquad [162]$$

Equation (161) gives, therefore, the effect of the curvatures of the ring, which for ordinary cases will be seen to be small. The resistance and inductance of the ring must, therefore, very approximately follow the same law of variation with the frequency as the straight wire.

The assumption of formula (65) that at high frequencies the magnetic field is symmetrical around the axis of the cross section of the ring is not strictly true. Actually, it will be a little stronger toward the axis of the ring, so that the amplitude of the current is slightly larger in that part of the cross section which is nearest the axis of the ring. This effect, however, will be extremely small and may be neglected.

We have, therefore, with great approximation

$$\varDelta L = -\pi a\left(1-\frac{4}{x}\frac{Z}{Y}\right)\left[1-\frac{1}{2}\frac{\rho^2}{a^2}\left(\log\frac{8a}{\rho}-\frac{1}{3}\right)\right] \qquad [163]$$

or, if terms in $\frac{\rho^2}{a^2}$ may be neglected,

$$\frac{\varDelta L}{L} = -\frac{\left(1-\frac{4}{x}\frac{Z}{Y}\right)}{4\log\frac{8a}{\rho}-7} \qquad [164]$$

$$\left(\frac{\varDelta L}{L}\right)_{x=\infty} = -\frac{1}{4\log\frac{8a}{\rho}-7} \qquad [165]$$

The values of $\left(\frac{\varDelta L}{L}\right)_{x=\infty}$ for various values of the determinative ratio $\frac{8a}{\rho}$ are tabulated in Table XXIII.

Neglecting the curvature, the change in resistance will be given by the same expression as for the straight wire, that is

$$\frac{R'}{R} = \frac{x}{2}\frac{W}{Y} \qquad [166]$$

The quantities $\frac{x}{2}\frac{W}{Y}$ and $\frac{4}{x}\frac{Z}{Y}$ are to be taken from Table XXII as before, the argument x being given by (156).

EXAMPLES ILLUSTRATING THE FORMULAS FOR HIGH FREQUENCY

EXAMPLE 82. STRAIGHT WIRE, VERY HIGH FREQUENCY

Let $f = 500000$ cycles per second, and $\therefore \lambda = 600$ meters
$l = 200$ cm
$\rho = 0.125$ cm.

If the wire is of copper, Table XXIV gives $x_0 = 15.146$ for a wire of $\rho = 0.1$ cm. We find, therefore, $x = 15.146 \times 1.25 = 18.932$. Entering Table XXII with this value of x, we find by interpolation, using second differences,

$$\frac{x}{2}\frac{W}{Y} = \frac{R'}{R} = 6.95035 \qquad \frac{4}{x}\frac{Z}{Y} = 0.14923$$

a slightly more accurate value of the latter may be found by making the interpolation for $\frac{Z}{Y}$.

$$\frac{2l}{\rho} = 3200 \text{ and } \therefore\ 4 \log \frac{2l}{\rho} - 3 = 29.284$$

By (154) $$\left(\frac{\Delta L}{L}\right)_{x=\infty} = -0.034148$$

The value found from Table XXIII by interpolation is 0.03416+.

Formula (147) gives therefore $$\frac{\Delta L}{L} = -0.034148(1 - 0.14923)$$
$$= -0.029052$$

By (145) $$\Delta L = -85.08 \text{ cm}$$

Recapitulating, the resistance at 500000 cycles per second is 6.95 times as great as with direct current, while the inductance is 85.08 cm or 2.9052 per cent less than the direct current value. This change of the inductance is 85.08 per cent of the possible change of 100 cm (0.034148 of the total inductance).

If the wire had been of manganin, for which the conductivity was one thirtieth of that of copper, the value of x becomes

$$x = 18.932 \times \sqrt{\frac{1}{30}} = 3.4566$$

and we find

$$\frac{R'}{R} = 1.47620 \qquad \frac{4}{x}\frac{Z}{Y} = .77255$$

$$1 - \frac{4}{x}\frac{Z}{Y} = .22745$$

The resistance is 1.47620 times the value at zero frequency, while the decrease of inductance is only 22.745 per cent of the total possible (0.034148), or 0.007767.

On the other hand, if the wire had been of iron (conductivity one-seventh of that of copper) and the permeability is assumed as low as 100

$$x = \sqrt{100}\sqrt{\frac{1}{7}}\, 18.932 = 71.556$$

$$\frac{R'}{R} = 25.551 \qquad \frac{4}{x}\frac{Z}{Y} = 0.039526 \qquad 1 - \frac{4}{x}\frac{Z}{Y} = .96047$$

By (146) $\qquad \left(\frac{\Delta L}{L}\right)_{x=\infty} = -\frac{100}{128.284} = -0.77950$

By (145) $\qquad (\Delta L)_{x=\infty} = -10000$ cm

and the actual changes are

$$\Delta L = -9605 \text{ cm}$$

$$\left(\frac{\Delta L}{L}\right) = -0.77950 \times .96047 = -0.7487$$

The influence of this relatively low permeability is enormous. The resistance is more than twenty-five times its direct current value, while the inductance is less than the direct current value by nearly 75 per cent of the latter, the maximum possible change with this permeability being about 78 per cent.

EXAMPLE 83. STRAIGHT WIRE—LOW FREQUENCY

If we consider the same wires as in the previous example, except that the frequency is assumed as only 1,000 per second.

Then for copper, $x = 0.6774 \times 1.25 = 0.84675$

$$\frac{R'}{R} = 1.00266 \qquad \frac{4}{x}\frac{Z}{Y} = 0.99867 \qquad 1 - \frac{4}{x}\frac{Z}{Y} = 0.00133$$

$$\Delta L = -0.133 \text{ cm}$$

$$\frac{\Delta L}{L} = -0.034148\,(.00133) = -0.000045$$

The resistance increase is only 0.266 per cent and the decrease of inductance is only about forty-five millionths of the total.

For manganin, $x = 0.84675\sqrt{\frac{1}{30}} = 0.1582$

By (151) $\quad \frac{R'}{R} = 1.0000030 \qquad \frac{4}{x}\frac{Z}{Y} = 1 - 0.0000015$

$$1 - \frac{4}{x}\frac{Z}{Y} = \quad 0.0000015$$

The increase in resistance is about three millionths and the decrease in inductance about five hundred-millionths of the direct current values.

For iron, with $\mu = 100$ as before

$$x = 0.84675\sqrt{100}\sqrt{\frac{1}{7}} = 3.2004$$

$$\frac{R'}{R} = 1.38516 \qquad \frac{4}{x}\frac{Z}{Y} = 0.81391 \qquad 1 - \frac{4}{x}\frac{Z}{Y} = 0.18609$$

$$\frac{\Delta L}{L} = -\frac{100}{128.284}(0.18609) = -0.14506$$

That is, the resistance increase is 38.5 per cent, the inductance decrease 14.5 per cent of the direct current values.

EXAMPLE 84. PARALLEL WIRES

Let us take wires of the same diameter and length as in examples 82 and 83 and consider the same frequencies. The values of $\frac{x}{2}\frac{W}{Y}$ and $\frac{4}{x}\frac{Z}{Y}$ will be the same as those in the cases corresponding in the previous examples. Further, assume that the distance between the centers of the wires is $d = 1.5$ cm.

Then $\qquad \frac{d}{\rho} = 12 \qquad 4 \log\frac{d}{\rho} + 1 = 10.9396$

∴ for nonmagnetic material

$$\left(\frac{\varDelta L}{L}\right)_{x=\infty}=-\frac{1}{10.9396}=-0.091412$$

as may also be found directly with sufficient precision from Table XXIII.

For iron wires, $\mu=100$

$$\left(\frac{\varDelta L}{L}\right)_{x=\infty}=-\frac{\mu}{4\log\frac{d}{\rho}+\mu}=-\frac{100}{109.94}=-0.9308.$$

The results for the cases treated in the previous examples are, therefore, for the parallel wires, as follows:

Material	Frequency	$\frac{R'}{R}$	$\frac{\varDelta L}{L}$
Copper	500000	6.9504	−0.077772
"	1000	1.00266	−0.000122
Manganin	500000	1.4762	−0.020792
"	1000	1.0000030	−0.00000014
Iron ($\mu=100$)	500000	25.551	−0.8940
"	1000	1.3852	−0.1732

The following table shows the effect of reducing the radius of cross section to $\rho=0.01$ cm

Material	Frequency	X	$\left(\frac{\varDelta L}{L}\right)_{x=\infty}$	$\frac{R'}{R}$	$\frac{\varDelta L}{L}$
Copper	500000	1.5146	0.047522	1.02682	−0.000636
"	1000	0.06774	"	1.0000011	-2.6×10^{-9}
Manganin	500000	0.27652	0.047522	1.000030	-7.2×10^{-7}
"	1000	0.01237	"	$1+1.2\times10^{-10}$	-3×10^{-12}
Iron ($\mu=100$)	500000	5.7245	0.83303	2.2974	0.4273
"	1000	0.25603	"	1.000022	-9.3×10^{-6}

EXAMPLE 85. CIRCULAR RING

Suppose the ring is of copper and that

$$\rho=0.1 \text{ cm},\quad a=20 \text{ cm},\quad \lambda=700 \text{ m}$$

Then from Table XXIV, $x=14.023$ and from Table XXII,

$$\frac{R'}{R}=5.2173,\quad 1-\frac{4}{x}\frac{Z}{Y}=0.79873$$

$$\log\frac{8a}{\rho} = \log 1600 = 7.37776$$

$$\frac{\rho}{a} = 0.005$$

By (162), $(\Delta L)_{x=\infty} = -20\pi = -62.83$ cm.

The correction term in (161) $= 1 - 0.0000880$

By (165) or Table XXIII,

$$\left(\frac{\Delta L}{L}\right)_{x=\infty} = -0.04442$$

and by (164),

$$\begin{aligned}\frac{\Delta L}{L} &= -0.04442 \times 0.79873\\ &= -0.03548\end{aligned}$$

WASHINGTON, January 1, 1911.

NOTE.

After the present paper had gone to press, a third formula for the mutual inductance of coaxial circles was published by Nagaoka (Tokyo Math. Phys., Soc. **6**, p. 10; 1911). This formula was given by Nagaoka in the following form:

$$M = 4\pi\sqrt{Aa}\left\{4\pi q^{\frac{3}{2}}\left(\frac{1 - 4q^3 + 9q^8 - \cdots\cdots}{1 - 3q^2 + 5q^6 - \cdots\cdots}\right)\right\}$$

The general term of the numerator being $(-1)^{n-1}n^2q^{n^2-1}$ and that of the denominator $(-1)^m(2m+1)q^{\frac{(2m+1)^2}{4}-\frac{1}{4}}$.

The quantity q is calculated from the modulus k_1', which is complementary to the modulus k_1 of formula (2). Using the same nomenclature as in section 1 of this collection we have

$$q = \frac{l}{2} + 2\left(\frac{l}{2}\right)^5 + 15\left(\frac{l}{2}\right)^9 + \cdots$$

$$l = \frac{1 - \sqrt{k_1'}}{1 + \sqrt{k_1'}} = \frac{k_1^2}{(1 + k_1')(1 + \sqrt{k_1'})^2}$$

$$k_1 = \frac{r_1 - r_2}{r_1 + r_2} = \frac{4Aa}{(r_1 + r_2)^2} \qquad k_1' = \frac{2\sqrt{r_1 r_2}}{r_1 + r_2}$$

$$r_1 = \sqrt{(A + a)^2 + d^2} \qquad r_2 = \sqrt{(A - a)^2 + d^2}$$

The general term of the above formula has been given for the sake of completeness. In general, however, the convergence is so rapid that all but the first terms are negligible.

As an example of the use of this formula, the calculation for the circles of examples 4 and 11 above is appended:

$$A = a = 25 \qquad d = 4$$

$$R_1 = \sqrt{2516} = 50.159744 \qquad R_2 = 4$$

$$k_1' = \frac{4\sqrt{2516}}{4+\sqrt{2516}} \qquad \sqrt{k_1'} = 0.72323683$$

$$\frac{l}{2} = 0.080303278 \qquad q = 0.080309959$$

$$1 - 4q^3 + 9q^8 = 0.99792812 \qquad 1 - 3q^2 + 5q^6 = 0.98065227$$

$$log\ q^{\frac{3}{4}} = \bar{1}.1785770 \qquad \therefore M = 606.0676\ cm$$

which agrees very closely with the value 606.0674 found in examples 4 and 11.

On expansion the above formula becomes

$$M = 4\pi\sqrt{Aa}\{4\pi q^{\frac{3}{4}}(1 + 3q^2 - 4q^3 + 9q^4 - 12q^5 + \cdots)\}$$

which suggests that the quantity q in this expression is equal to the square of the corresponding quantity in formula (8) above. The truth of this proposition may be established by expressing Landen's transformation in terms of q functions.

As regards numerical calculation, therefore, this last formula of Nagaoka is entirely equivalent to his earlier formula (8).

APPENDIX

TABLES OF CONSTANTS AND FUNCTIONS USEFUL IN THE CALCULATION OF MUTUAL AND SELF-INDUCTANCE

TABLE I

Maxwell's Table of Values of $\text{Log}\,\frac{M}{4\pi\sqrt{Aa}}=\left[\left(\frac{2}{k}-k\right)F-\frac{2}{k}E\right]$

(For use with Formula (1))

$\gamma=\sin^{-1}\frac{\sqrt{r_1^2-r_2^2}}{r_1}$	$\text{Log}\frac{M}{4\pi\sqrt{Aa}}$	Δ_1	$\gamma=\sin^{-1}\frac{\sqrt{r_1^2-r_2^2}}{r_1}$	$\text{Log}\frac{M}{4\pi\sqrt{Aa}}$	Δ_1
60° 0′	$\bar{1}$.499 4780	2 7868	65° 0′	$\bar{1}$.637 6633	2 7508
6′	$\bar{1}$.502 2648	2 7854	6′	$\bar{1}$.640 4141	2 7508
12′	$\bar{1}$.505 0502	2 7840	12′	$\bar{1}$.643 1649	2 7507
18′	$\bar{1}$.507 8342	2 7828	18′	$\bar{1}$.645 9156	2 7507
24′	$\bar{1}$.510 6170	2 7816	24′	$\bar{1}$.648 6663	2 7507
30′	$\bar{1}$.513 3986	2 7803	30′	$\bar{1}$.651 4170	2 7509
36′	$\bar{1}$.516 1789	2 7790	36′	$\bar{1}$.654 1679	2 7510
42′	$\bar{1}$.518 9579	2 7778	42′	$\bar{1}$.656 9189	2 7512
48′	$\bar{1}$.521 7357	2 7765	48′	$\bar{1}$.659 6701	2 7514
54′	$\bar{1}$.524 5122	2 7753	54′	$\bar{1}$.662 4215	2 7516
61° 0′	$\bar{1}$.527 2875	2 7743	66° 0′	$\bar{1}$.665 1731	2 7519
6′	$\bar{1}$.530 0618	2 7734	6′	$\bar{1}$.667 9250	2 7522
12′	$\bar{1}$.532 8352	2 7725	12′	$\bar{1}$.670 6772	2 7524
18′	$\bar{1}$.535 6077	2 7715	18′	$\bar{1}$.673 4296	2 7528
24′	$\bar{1}$.538 3792	2 7705	24′	$\bar{1}$.676 1824	2 7532
30′	$\bar{1}$.541 1497	2 7694	30′	$\bar{1}$.678 9356	2 7535
36′	$\bar{1}$.543 9191	2 7683	36′	$\bar{1}$.681 6891	2 7539
42′	$\bar{1}$.546 6874	2 7672	42′	$\bar{1}$.684 4430	2 7543
48′	$\bar{1}$.549 4546	2 7663	48′	$\bar{1}$.687 1973	2 7548
54′	$\bar{1}$.552 2209	2 7654	54′	$\bar{1}$.689 9521	2 7553
62° 0′	$\bar{1}$.554 9863	2 7645	67° 0′	$\bar{1}$.692 7074	2 7561
6′	$\bar{1}$.557 7508	2 7637	6′	$\bar{1}$.695 4635	2 7567
12′	$\bar{1}$.560 5145	2 7629	12′	$\bar{1}$.698 2202	2 7573
18′	$\bar{1}$.563 2774	2 7622	18′	$\bar{1}$.700 9775	2 7580
24′	$\bar{1}$.566 0396	2 7615	24′	$\bar{1}$.703 7355	2 7587
30′	$\bar{1}$.568 8011	2 7607	30′	$\bar{1}$.706 4942	2 7595
36′	$\bar{1}$.571 5618	2 7598	36′	$\bar{1}$.709 2537	2 7603
42′	$\bar{1}$.574 3216	2 7589	42′	$\bar{1}$.712 0140	2 7610
48′	$\bar{1}$.577 0805	2 7582	48′	$\bar{1}$.714 7750	2 7619
54′	$\bar{1}$.579 8387	2 7575	54′	$\bar{1}$.717 5369	2 7628
63° 0′	$\bar{1}$.582 5962	2 7570	68° 0′	$\bar{1}$.720 2997	2 7637
6′	$\bar{1}$.585 3532	2 7567	6′	$\bar{1}$.723 0634	2 7647
12′	$\bar{1}$.588 1099	2 7563	12′	$\bar{1}$.725 8281	2 7656
18′	$\bar{1}$.590 8662	2 7559	18′	$\bar{1}$.728 5937	2 7667
24′	$\bar{1}$.593 6221	2 7555	24′	$\bar{1}$.731 3604	2 7679
30′	$\bar{1}$.596 3776	2 7549	30′	$\bar{1}$.734 1283	2 7689
36′	$\bar{1}$.599 1325	2 7543	36′	$\bar{1}$.736 8972	2 7701
42′	$\bar{1}$.601 8868	2 7537	42′	$\bar{1}$.739 6673	2 7713
48′	$\bar{1}$.604 6405	2 7533	48′	$\bar{1}$.742 4386	2 7725
54′	$\bar{1}$.607 3938	2 7530	54′	$\bar{1}$.745 2111	2 7737
64° 0′	$\bar{1}$.610 1468	2 7527	69° 0′	$\bar{1}$.747 9848	2 7749
6′	$\bar{1}$.612 8995	2 7524	6′	$\bar{1}$.750 7597	2 7763
12′	$\bar{1}$.615 6519	2 7521	12′	$\bar{1}$.753 5360	2 7778
18′	$\bar{1}$.618 4040	2 7519	18′	$\bar{1}$.756 3138	2 7791
24′	$\bar{1}$.621 1559	2 7516	24′	$\bar{1}$.759 0929	2 7806
30′	$\bar{1}$.623 9075	2 7514	30′	$\bar{1}$.761 8735	2 7821
36′	$\bar{1}$.626 6589	2 7513	36′	$\bar{1}$.764 6556	2 7836
42′	$\bar{1}$.629 4102	2 7512	42′	$\bar{1}$.767 4392	2 7853
48′	$\bar{1}$.632 1614	2 7510	48′	$\bar{1}$.770 2245	2 7871
54′	$\bar{1}$.634 9124	2 7509	54′	$\bar{1}$.773 0116	2 7888
65° 0′	$\bar{1}$.637 6633	2 7508	70° 0′	$\bar{1}$.775 8004	2 7904

TABLE I—Continued

$\gamma=\sin^{-1}\frac{\sqrt{r_1^2-r_2^2}}{r_1}$	$\text{Log}\frac{M}{4\pi\sqrt{Aa}}$	Δ_1	$\gamma=\sin^{-1}\frac{\sqrt{r_1^2-r_2^2}}{r_1}$	$\text{Log}\frac{M}{4\pi\sqrt{Aa}}$	Δ_1
70° 0′	$\bar{1}$.775 8004	2 7904	75° 0′	$\bar{1}$.918 5141	2 9472
6′	$\bar{1}$.778 5908	2 7920	6′	$\bar{1}$.921 4613	2 9522
12′	$\bar{1}$.781 3828	2 7938	12′	$\bar{1}$.924 4135	2 9572
18′	$\bar{1}$.734 1766	2 7956	18′	$\bar{1}$.927 3707	2 9623
24′	$\bar{1}$.786 9722	2 7975	24′	$\bar{1}$.930 3330	2 9676
30′	$\bar{1}$.789 7697	2 7995	30′	$\bar{1}$.933 3006	2 9729
36′	$\bar{1}$.792 5692	2 8017	36′	$\bar{1}$.936 2735	2 9783
42′	$\bar{1}$.795 3709	2 8037	42′	$\bar{1}$.939 2518	2 9838
48′	$\bar{1}$.798 1746	2 8056	48′	$\bar{1}$.942 2356	2 9895
54′	$\bar{1}$.800 9802	2 8078	54′	$\bar{1}$.945 2251	2 9951
71° 0′	$\bar{1}$.803 7880	2 8100	76° 0′	$\bar{1}$.948 2202	3 0007
6′	$\bar{1}$.806 5980	2 8124	6′	$\bar{1}$.951 2209	3 0066
12′	$\bar{1}$.809 4104	2 8148	12′	$\bar{1}$.954 2275	3 0127
18′	$\bar{1}$.812 2252	2 8172	18′	$\bar{1}$.957 2402	3 0188
24′	$\bar{1}$.815 0424	2 8195	24′	$\bar{1}$.960 2590	3 0251
30′	$\bar{1}$.817 8619	2 8220	30′	$\bar{1}$.963 2841	3 0316
36′	$\bar{1}$.820 6839	2 8245	36′	$\bar{1}$.966 3157	3 0380
42′	$\bar{1}$.823 5084	2 8270	42′	$\bar{1}$.969 3537	3 0446
48′	$\bar{1}$.826 3354	2 8297	48′	$\bar{1}$.972 3983	3 0514
54′	$\bar{1}$.829 1651	2 8323	54′	$\bar{1}$.975 4497	3 0583
72° 0′	$\bar{1}$.831 9974	2 8349	77° 0′	$\bar{1}$.978 5080	3 0652
6′	$\bar{1}$.834 8323	2 8377	6′	1.981 5731	3 0723
12′	$\bar{1}$.837 6700	2 8406	12′	$\bar{1}$.984 6454	3 0795
18′	$\bar{1}$.840 5106	2 8435	18′	$\bar{1}$.987 7249	3 0869
24′	$\bar{1}$.843 3541	2 8464	24′	$\bar{1}$.990 8118	3 0944
30′	$\bar{1}$.846 2005	2 8494	30′	$\bar{1}$.993 9062	3 1020
36′	$\bar{1}$.849 0499	2 8525	36′	$\bar{1}$.997 0082	3 1099
42′	$\bar{1}$.851 9024	2 8556	42′	0.000 1181	3 1178
48′	$\bar{1}$.854 7580	2 8588	48′	0.003 2359	3 1259
54′	$\bar{1}$.857 6168	2 8620	54′	0.006 3618	3 1341
73° 0′	$\bar{1}$.860 4788	2 8653	78° 0′	0.009 4959	3 1426
6′	$\bar{1}$.863 3441	2 8688	6′	0.012 6385	3 1511
12′	$\bar{1}$.866 2129	2 8723	12′	0.015 7896	3 1598
18′	$\bar{1}$.869 0852	2 8759	18′	0.018 9494	3 1687
24′	$\bar{1}$.871 9611	2 8795	24′	0.022 1181	3 1778
30′	$\bar{1}$.874 8406	2 8831	30′	0.025 2959	3 1871
36′	$\bar{1}$.877 7237	2 8869	36′	0.028 4830	3 1964
42′	$\bar{1}$.880 6106	2 8907	42′	0.031 6794	3 2061
48′	$\bar{1}$.883 5013	2 8946	48′	0.034 8855	3 2159
54′	$\bar{1}$.886 3959	2 8986	54′	0.038 1014	3 2258
74° 0′	$\bar{1}$.889 2945	2 9025	79° 0′	0.041 3272	3 2360
6′	$\bar{1}$.892 1970	2 9066	6′	0.044 5633	3 2465
12′	$\bar{1}$.895 1036	2 9108	12′	0.047 8098	3 2570
18′	$\bar{1}$.898 0144	2 9151	18′	0.051 0668	3 2679
24′	$\bar{1}$.900 9295	2 9194	24′	0.054 3347	3 2789
30′	$\bar{1}$.903 8489	2 9239	30′	0.057 6136	3 2901
36′	$\bar{1}$.906 7728	2 9284	36′	0.060 9037	3 3016
42′	$\bar{1}$.909 7012	2 9329	42′	0.064 2053	3 3132
48′	$\bar{1}$.912 6341	2 9376	48′	0.067 5185	3 3252
54′	$\bar{1}$.915 5717	2 9424	54′	0.070 8437	3 3375
75° 0′	$\bar{1}$.918 5141	2 9472	80° 0′	0.074 1812	3 3500

TABLE I—Continued

$\gamma=\sin^{-1}\frac{\sqrt{r_1^2-r_2^2}}{r_1}$	$\text{Log}\frac{M}{4\pi\sqrt{Aa}}$	Δ_1	$\gamma=\sin^{-1}\frac{\sqrt{r_1^2-r_2^2}}{r_1}$	$\text{Log}\frac{M}{4\pi\sqrt{Aa}}$	Δ_1
80° 0′	0.074 1812	3 3500	85° 0′	0.265 4154	4 6004
6′	0.077 5312	3 3628	6′	0.270 0156	4 6499
12′	0.080 8940	3 3760	12′	0.274 6655	4 7015
18′	0.084 2700	3 3892	18′	0.279 3670	4 7553
24′	0.087 6592	3 4027	24′	0.284 1223	4 8109
30′	0.091 0619	3 4165	30′	0.288 9332	4 8689
36′	0.094 4784	3 4307	36′	0.293 8021	4 9293
42′	0.097 9091	3 4452	42′	0.298 7314	4 9924
48′	0.101 3543	3 4601	48′	0.303 7238	5 0585
54′	0.104 8144	3 4752	54′	0.308 7823	5 1274
81° 0′	0.108 2896	3 4906	86° 0′	0.313 9097	5 1995
6′	0.111 7802	3 5064	6′	0.319 1092	5 2751
12′	0.115 2866	3 5226	12′	0.324 3843	5 3544
18′	0.118 8092	3 5392	18′	0.329 7387	5 4375
24′	0.122 3484	3 5561	24′	0.335 1762	5 5250
30′	0.125 9045	3 5735	30′	0.340 7012	5 6172
36′	0.129 4780	3 5912	36′	0.346 3184	5 7143
42′	0.133 0692	3 6094	42′	0.352 0327	5 8168
48′	0.136 6786	3 6280	48′	0.357 8495	5 9254
54′	0.140 3066	3 6470	54′	0.363 7749	6 0404
82° 0′	0.143 9536	3 6667	87° 0′	0.369 8154	6 1624
6′	0.147 6203	3 6869	6′	0.375 9777	6 2923
12′	0.151 3072	3 7076	12′	0.382 2700	6 4306
18′	0.155 0148	3 7287	18′	0.388 7006	6 5786
24′	0.158 7435	3 7503	24′	0.395 2792	6 7370
30′	0.162 4938	3 7722	30′	0.402 0162	6 9072
36′	0.166 2660	3 7949	36′	0.408 9234	7 0904
42′	0.170 0609	3 8183	42′	0.416 0138	7 2884
48′	0.173 8792	3 8425	48′	0.423 3022	7 5031
54′	0.177 7217	3 8673	54′	0.430 8053	7 7373
83° 0′	0.181 5890	3 8926	88° 0′	0.438 5417	7 9921
6′	0.185 4816	3 9185	6′	0.446 5341	8 2723
12′	0.189 4001	3 9452	12′	0.454 8064	8 5816
18′	0.193 3453	3 9728	18′	0.463 3880	8 9247
24′	0.197 3181	4 0013	24′	0.472 3127	9 3079
30′	0.201 3194	4 0308	30′	0.481 6206	9 7389
36′	0.205 3502	4 0606	36′	0.491 3595	10 2275
42′	0.209 4108	4 0915	42′	0.501 5870	10 7868
48′	0.213 5023	4 1236	48′	0.512 3738	11 4341
54′	0.217 6259	4 1565	54′	0.523 8079	12 1932
84° 0′	0.221 7824	4 1904	89° 0′	0.536 0011	13 0958
6′	0.225 9728	4 2255	6′	0.549 0969	14 1917
12′	0.230 1983	4 2617	12′	0.563 2886	15 5520
18′	0.234 4600	4 2991	18′	0.578 8406	17 2914
24′	0.238 7591	4 3379	24′	0.596 1320	19 6050
30′	0.243 0970	4 3778	30′	0.615 7370	22 8537
36′	0.247 4748	4 4192	36′	0.638 5907	27 7976
42′	0.251 8940	4 4621	42′	0.666 3883	36 3882
48′	0.256 3561	4 5065	48′	0.702 7765	55 9176
54′	0.260 8626	4 5526	54′	0.758 6941	
85° 0′	0.265 4154	4 6004			

The above table has been recalculated and some of the values corrected in the last place. The values given are sufficiently accurate to give M within one part in a million.

TABLE II

Giving the Values of Log F and Log E as Functions of tan γ. (*See p. 20*)

tan γ	Log F	F	Log E	E
0.1	0.1971 996	1.5747 065	0.1950 415	1.5669 007
0.2	0.2003 678	1.5862 361	0.1918 928	1.5555 817
0.3	0.2054 261	1.6048 192	0.1869 144	1.5378 514
0.4	0.2120 849	1.6296 146	0.1804 536	1.5151 429
0.5	0.2200 096	1.6596 236	0.1729 048	1.4890 346
0.6	0.2288 634	1.6938 051	0.1646 557	1.4610 185
0.7	0.2383 385	1.7311 652	0.1560 492	1.4323 502
0.8	0.2481 728	1.7708 135	0.1473 640	1.4039 900
0.9	0.2581 561	1.8119 912	0.1388 116	1.3766 121
1.0	0.2681 272	1.8540 745	0.1305 409	1.3506 441
1.5	0.3147 473	2.0641 787	0.0955 992	1.2462 329
2.0	0.3535 711	2.2572 057	0.0713 258	1.1784 897
2.5	0.3852 192	2.4278 352	0.0547 850	1.1344 491
3.0	0.4112 984	2.5780 917	0.0432 738	1.1047 748
4.0	0.4518 237	2.8302 429	0.0289 324	1.0688 885
5.0	0.4821 752	3.0351 154	0.0207 426	1.0489 205
7.5	0.5341 061	3.4206 300	0.0109 567	1.0255 497
10.0	0.5682 672	3.7005 581	0.0068 338	1.0158 598
12.5	0.5932 708	3.9198 622	0.0047 004	1.0108 819

TABLE III

Values of the Constant K as Functions of x/A and a/A

(For use in Formula (57))

x/A = / a/A	.50	.75	1	1.25	1.50	1.75	2
0.50	9.39283	12.30385	14.27982	15.62795	16.56549	17.23299	17.71973
0.55	9.52044	12.40135	14.34594	15.67140	16.59411	17.25215	17.73283
0.60	9.66358	12.50816	14.41766	15.71837	16.62503	17.27286	17.74701
0.65	9.82296	12.62412	14.49474	15.76867	16.65813	17.29504	17.76221
0.70	9.99921	12.74897	14.57688	15.82212	16.69330	17.31865	17.77841
0.75	10.19272	12.88232	14.66377	15.87850	16.73039	17.34357	17.79554

TABLE IV

Values of the Constant Q in Formula (74), $L_s = n^2aQ$

For the self-inductance of a single-layer winding on a solenoid; n is the whole number of turns of wire in the winding and a is the mean radius. The corrections by Tables VII and VIII must be made to get L from L_s as usual. (See p. 122.)

In the following table $2a$ is the diameter, b is the length, and therefore $2a/b = \tan\gamma$. (See Fig. 33.)

$2\frac{a}{b} = \tan\gamma$	Q	$2\frac{a}{b} = \tan\gamma$	Q
0.20	3.63240	1.80	19.57938
0.30	5.23368	2.00	20.74631
0.40	6.71017	2.20	21.82049
0.50	8.07470	2.40	22.81496
0.60	9.33892	2.60	23.74013
0.70	10.51349	2.80	24.60482
0.80	11.60790	3.00	25.41613
0.90	12.63059	3.20	26.18009
1.00	13.58892	3.40	26.90177
1.20	15.33799	3.60	27.58548
1.40	16.89840	3.80	28.23494
1.60	18.30354	4.00	28.85335

For an explanation of the above formula, see page 118.

TABLE V

Constants A and B for Strasser's Formula (82)

$$A=2\log_e[(n-1)!(n-2)!\cdot\cdot\cdot\cdot 1]$$
$$B=3[(n-2)2^2\log_e 2+(n-3)3^2\log_e 3+\cdot\cdot\cdot(n-1)^2\log_e(n-1)]$$

n	A	B	n	A	B
1	0	0	16	354.396	35693
2	0	0	17	415.739	46775
3	1.38629	8.318	18	482.75	60314
4	4.96981	46.298	19	555.54	76662
5	11.3259	150.82	20	634.22	96198
6	20.9009	376.05	21	718.89	119330
7	34.0594	794.79	22	809.65	146490
8	51.1097	1499.58	23	906.59	178140
9	72.3189	2603.62	24	1009.81	214760
10	97.9226	4241.59	25	1119.38	256880
11	128.131	6570.33	26	1235.38	305030
12	163.136	9769.51	27	1357.91	359790
13	203.110	14042.2	28	1487.02	421750
14	248.215	19615.3	29	1622.80	491560
15	298.597	26740.1	30	1765.32	569860

We have recently recomputed Strasser's constants, finding several errors which are corrected here.

TABLE VI

Table of Constants for Stefan's Formula (90)

b/c or c/b	y_1	c/b	y_2	b/c	y_2
0.00	0.50000	0.00	0.1250	0.00	Infinite
.05	.54899	.05	.1269	.05	239.43
.10	.59243	.10	.1325	.10	60.231
.15	.63102	.15	.1418	.15	27.020
.20	.66520	.20	.1548	.20	15.378
.25	.69532	.25	.1714	.25	9.9765
.30	.72172	.30	.1916	.30	7.0327
.35	.74469	.35	.2152	.35	5.2502
.40	.76454	.40	.2423	.40	4.0876
.45	.78154	.45	.2728	.45	3.2861
.50	.79600	.50	.3066	.50	2.7093
.55	.80815	.55	.3437	.55	2.2798
.60	.81823	.60	.3839	.60	1.9509
.65	.82648	.65	.4273	.65	1.6931
.70	.83311	.70	.4739	.70	1.4871
.75	.83831	.75	.5234	.75	1.3198
.80	.84225	.80	.5760	.80	1.1819
.85	.84509	.85	.6317	.85	1.0669
.90	.84697	.90	.6902	.90	0.9698
.95	.84801	.95	.7518	.95	0.8872
1.00	.84834	1.00	.8162	1.00	0.8162

TABLE VII

Values of Correction Term A, Depending on the Ratio $\frac{d}{D}$ of the Diameters of Bare and Covered Wire on the Single Layer Coil

(For use in Formula (80))

$$A=\log_e\left(1.7452\frac{d}{D}\right)$$

$\frac{d}{D}$	A	Δ_1	$\frac{d}{D}$	A	Δ_1	$\frac{d}{D}$	A	Δ_1	Δ_2
1.00	0.5568	−100	0.75	0.2691	−134	0.50	−0.1363	−202	− 4
.99	.5468	−101	.74	.2557	−136	.49	− .1565	−206	− 5
.98	.5367	−103	.73	.2421	−138	.48	− .1771	−211	− 4
.97	.5264	−104	.72	.2283	−140	.47	− .1982	−215	− 4
.96	.5160	−105	.71	.2143	−142	.46	− .2197	−219	− 6
0.95	0.5055	−106	0.70	0.2001	−144	0.45	−0.2416	−225	− 5
.94	.4949	−107	.69	.1857	−146	.44	− .2641	−230	− 5
.93	.4842	−108	.68	.1711	−148	.43	− .2871	−235	− 6
.92	.4734	−109	.67	.1563	−150	.42	− .3106	−241	− 6
.91	.4625	−110	.66	.1413	−152	.41	− .3347	−247	− 6
0.90	0.4515	−112	0.65	0.1261	−155	0.40	−0.3594	−253	− 7
.89	.4403	−113	.64	.1106	−157	.39	− .3847	−260	− 7
.88	.4290	−114	.63	.0949	−160	.38	− .4107	−267	− 7
.87	.4176	−116	.62	.0789	−163	.37	− .4374	−274	− 7
.86	.4060	−117	.61	.0626	−166	.36	− .4648	−281	− 9
0.85	0.3943	−118	0.60	0.0460	−168	0.35	−0.4929	−290	− 9
.84	.3825	−120	.59	.0292	−171	.34	− .5219	−299	− 9
.83	.3705	−121	.58	+ .0121	−174	.33	− .5518	−308	− 9
.82	.3584	−123	.57	− .0053	−177	.32	− .5826	−317	− 9
.81	.3461	−124	.56	− .0230	−180	.31	− .6143	−328	−11
0.80	0.3337	−126	0.55	−0.0410	−184	0.30	−0.6471	−339	−12
.79	.3211	−127	.54	− .0594	−187	.29	− .6810	−351	−13
.78	.3084	−129	.53	− .0781	−190	.28	− .7161	−364	−13
.77	.2955	−131	.52	− .0971	−194	.27	− .7525	−377	−15
.76	.2824	−133	.51	− .1165	−198	.26	− .7902	−392	−16
0.75	0.2691		0.50	−0.1363		0.25	−0.8294		

TABLE VII—Continued

$\frac{d}{D}$	A	Δ_1	Δ_2	$\frac{d}{D}$	A	Δ_1	Δ_2
0.25	−0.8294	−408	− 18	0.10	−1.7457	−1054	− 124
.24	− .8702	−426	− 19	.09	−1.8511	−1178	− 157
.23	− .9128	−445	− 20	.08	−1.9689	−1335	− 206
.22	− .9573	−465	− 23	.07	−2.1024	−1541	− 283
.21	−1.0038	−488	− 25	.06	−2.2565	−1824	− 407
0.20	−1.0526	−513	− 28	0.05	−2.4389	−2231	− 646
.19	−1.1039	−541	− 30	.04	−2.6620	−2877	−1177
.18	−1.1580	−571	− 35	.03	−2.9497	−4054	−2878
.17	−1.2151	−606	− 39	.02	−3.3551	−6932	
.16	−1.2757	−645	− 45	.01	−4.0483		
0.15	−1.3402	−690	− 51				
.14	−1.4092	−741	− 60				
.13	−1.4833	−801	− 70				
.12	−1.5634	−870	− 83				
.11	−1.6504	−953	−101				
0.10	−1.7457						

TABLE VIII

Values of the Correction Term B, Depending on the Number of Turns of Wire on the Single Layer Coil

(For use in Formula (80))

$$B = \frac{2}{n} \sum_{1}^{n-1} m \log_e \frac{m}{R_m}$$

where R_m is geometric mean distance of the sections of the current sheet whose centers coincide with those of the wires. (See this Bulletin, 2, p. 168, equat. (11): 1906.)

Number of Turns=n	B	Number of Turns=n	B
1	0.0000	50	0.3186
2	.1137	60	.3216
3	.1663	70	.3239
4	.1973	80	.3257
5	.2180	90	.3270
6	.2329	100	.3280
7	.2443	125	.3298
8	.2532	150	.3311
9	.2604	175	.3321
10	.2664	200	.3328
15	.2857	300	.3343
20	.2974	400	.3351
25	.3042	500	.3356
30	.3083	600	.3359
35	.3119	700	.3361
40	.3148	800	.3363
45	.3169	900	.3364
50	.3186	1000	.3365

TABLE IX

Value of the Constant A_s as a Function of t/a, t being the Depth of the Winding and a the Mean Radius

$$A_s = 0.6949 - \frac{t^2}{96a^2}\left(\log_e \frac{8a}{t} + 2.76\right)$$

(For use in Formula (91))

t/a	A_s
0	0.6949
0.10	0.6942
0.15	0.6933
0.20	0.6922
0.25	0.6909

TABLE X

Values of the Correction Term B_s depending on the Number of Turns of Square Conductor on the equivalent Single Layer Coil shown in Fig. 40

(For use in Formula (91))

$$B_s = \frac{2}{n} \sum_{1}^{n-1} \left(m \log \frac{R'_m}{R_m} \right).$$

where R'_m=geom. mean distance for the two squares
R_m= " " " " " elements of the current sheet. (See this Bulletin, 4, p. 373; 1907.)

Number of Turns b/c	B_s	Number of Turns b/c	B_s	Number of Turns b/c	B_s
1	0.0000	11	0.2844	21	0.3116
2	.1202	12	.2888	22	.3131
3	.1753	13	.2927	23	.3145
4	.2076	14	.2961	24	.3157
5	.2292	15	.2991	25	.3169
6	.2446	16	.3017	26	.3180
7	.2563	17	.3041	27	.3190
8	.2656	18	.3062	28	.3200
9	.2730	19	.3082	29	.3209
10	.2792	20	.3099	30	.3218

TABLE XI

Table of Napierian Logarithms to Nine Decimal Places for Numbers from 1 to 100

1	0.000 000 000	51	3.931 825 633
2	0.693 147 181	52	3.951 243 719
3	1.098 612 289	53	3.970 291 914
4	1.386 294 361	54	3.988 984 047
5	1.609 437 912	55	4.007 333 185
6	1.791 759 469	56	4.025 351 691
7	1.945 910 149	57	4.043 051 268
8	2.079 441 542	58	4.060 443 011
9	2.197 224 577	59	4.077 537 444
10	2.302 585 093	60	4.094 344 562
11	2.397 895 273	61	4.110 873 864
12	2.484 906 650	62	4.127 134 385
13	2.564 949 357	63	4.143 134 726
14	2.639 057 330	64	4.158 883 083
15	2.708 050 201	65	4.174 387 270
16	2.772 588 722	66	4.189 654 742
17	2.833 213 344	67	4.204 692 619
18	2.890 371 758	68	4.219 507 705
19	2.944 438 979	69	4.234 106 505
20	2.995 732 274	70	4.248 495 242
21	3.044 522 438	71	4.262 679 877
22	3.091 042 453	72	4.276 666 119
23	3.135 494 216	73	4.290 459 441
24	3.178 053 830	74	4.304 065 093
25	3.218 875 825	75	4.317 488 114
26	3.258 096 538	76	4.330 733 340
27	3.295 836 866	77	4.343 805 422
28	3.332 204 510	78	4.356 708 827
29	3.367 295 830	79	4.369 447 852
30	3.401 197 382	80	4.382 026 635
31	3.433 987 204	81	4.394 449 155
32	3.465 735 903	82	4.406 719 247
33	3.496 507 561	83	4.418 840 608
34	3.526 360 525	84	4.430 816 799
35	3.555 348 061	85	4.442 651 256
36	3.583 518 938	86	4.454 347 296
37	3.610 917 913	87	4.465 908 119
38	3.637 586 160	88	4.477 336 814
39	3.663 561 646	89	4.488 636 370
40	3.688 879 454	90	4.499 809 670
41	3.713 572 067	91	4.510 859 507
42	3.737 669 618	92	4.521 788 577
43	3.761 200 116	93	4.532 599 493
44	3.784 189 634	94	4.543 294 782
45	3.806 662 490	95	4.553 876 892
46	3.828 641 396	96	4.564 348 191
47	3.850 147 602	97	4.574 710 979
48	3.871 201 011	98	4.584 967 479
49	3.891 820 298	99	4.595 119 850
50	3.912 023 005	100	4.605 170 186

$\log 1525 = \log 25 + \log 61$; $\log 9.8 = \log 98 - \log 10$, etc.

TABLE XII

Values of F and E

The following table of elliptic integrals of the first and second kind is taken from Legendre's *Traité des Fonctions Elliptiques*, Volume 2, Table VIII:

	F	Δ_1	Δ_2		E	Δ_1	Δ_2
0°	1.570 796	120	239	0°	1.570 796	− 120	−239
1	1.570 916	359	240	1	1.570 677	− 359	−239
2	1.571 275	599	240	2	1.570 318	− 598	−239
3	1.571 874	839	241	3	1.569 720	− 836	−238
4	1.572 712	1 080	241	4	1.568 884	−1 075	−238
5	1.573 792	1 321	243	5	1.567 809	−1 312	−237
6	1.575 114	1 564	244	6	1.566 497	−1 549	−236
7	1.576 678	1 808	246	7	1.564 948	−1 785	−235
8	1.578 486	2 054	247	8	1.563 162	−2 020	−234
9	1.580 541	2 302	249	9	1.561 142	−2 255	−233
10	1.582 843	2 551	252	10	1.558 887	−2 487	−232
11	1.585 394	2 803	254	11	1.556 400	−2 719	−230
12	1.588 197	3 057	257	12	1.553 681	−2 949	−228
13	1.591 254	3 314	260	13	1.550 732	−3 177	−227
14	1.594 568	3 574	263	14	1.547 554	−3 404	−225
15	1.598 142	3 836	266	15	1.544 150	−3 629	−223
16	1.601 978	4 103	270	16	1.540 521	−3 852	−221
17	1.606 081	4 373	274	17	1.536 670	−4 073	−218
18	1.610 454	4 647	278	18	1.532 597	−4 291	−216
19	1.615 101	4 925	283	19	1.528 306	−4 507	−214
20	1.620 026	5 208	288	20	1.523 799	−4 721	−211
21	1.625 234	5 495	293	21	1.519 079	−4 932	−208
22	1.630 729	5 788	298	22	1.514 147	−5 140	−205
23	1.636 517	6 087	304	23	1.509 007	−5 345	−202
24	1.642 604	6 391	311	24	1.503 662	−5 547	−199
25	1.648 995	6 702	317	25	1.498 115	−5 746	−196
26	1.655 697	7 019	324	26	1.492 368	−5 942	−192
27	1.662 716	7 343	332	27	1.486 427	−6 134	−189
28	1.670 059	7 675	340	28	1.480 293	−6 323	−185
29	1.677 735	8 015	349	29	1.473 970	−6 508	−181
30	1.685 750	8 364	358	30	1.467 462	−6 689	−177
31	1.694 114	8 722	367	31	1.460 774	−6 866	−173
32	1.702 836	9 089	377	32	1.453 908	−7 039	−168
33	1.711 925	9 466	388	33	1.446 869	−7 207	−164
34	1.721 391	9 854	400	34	1.439 662	−7 371	−159
35	1.731 245	10 254	412	35	1.432 291	−7 531	−155
36	1.741 499	10 666	425	36	1.424 760	−7 685	−150
37	1.752 165	11 091	439	37	1.417 075	−7 835	−145
38	1.763 256	11 530	453	38	1.409 240	−7 980	−140
39	1.774 786	11 983	469	39	1.401 260	−8 120	−134
40	1.786 770	12 452	486	40	1.393 140	−8 254	−129
41	1.799 222	12 938	504	41	1.384 886	−8 382	−123
42	1.812 160	13 442	523	42	1.376 504	−8 505	−117
43	1.825 602	13 965	543	43	1.367 999	−8 622	−111
44	1.839 567	14 508	565	44	1.359 377	−8 733	−105
45	1.854 075	15 073	588	45	1.350 644	−8 838	− 98

TABLE XII—Continued

	F	Δ_1	Δ_2		E	Δ_1	Δ_2
45°	1.854 075	15 073	588	45°	1.350 644	−8 838	−98
46	1.869 148	15 661	613	46	1.341 806	−8 936	−92
47	1.884 809	16 274	640	47	1.332 870	−9 028	−85
48	1.901 083	16 914	669	48	1.323 842	−9 113	−78
49	1.917 997	17 584	700	49	1.314 729	−9 190	−71
50	1.935 581	18 284	735	50	1.305 539	−9 261	−63
51	1.953 865	19 017	770	51	1.296 278	−9 324	−56
52	1.972 882	19 787	809	52	1.286 954	−9 380	−48
53	1.992 670	20 597	852	53	1.277 574	−9 427	−40
54	2.013 266	21 449	898	54	1.268 147	−9 467	−31
55	2.034 715	22 347	949	55	1.258 680	−9 498	−22
56	2.057 062	23 296	1 004	56	1.249 182	−9 520	−14
57	2.080 358	24 300	1 064	57	1.239 661	−9 534	− 4
58	2.104 658	25 364	1 130	58	1.230 127	−9 538	+ 5
59	2.130 021	26 494	1 203	59	1.220 589	−9 533	+15
60	2.156 516	27 698	1 284	60	1.211 056	−9 518	+25
61	2.184 213	28 982	1 373	61	1.201 538	−9 492	36
62	2.213 195	30 355	1 472	62	1.192 046	−9 457	47
63	2.243 549	31 827	1 583	63	1.182 589	−9 410	58
64	2.275 376	33 410	1 708	64	1.173 180	−9 351	70
65	2.308 787	35 118	1 848	65	1.163 828	−9 281	82
66	2.343 905	36 965	2 006	66	1.154 547	−9 199	95
67	2.380 870	38 971	2 186	67	1.145 348	−9 104	109
68	2.419 842	41 158	2 393	68	1.136 244	−8 995	123
69	2.460 999	43 551	2 631	69	1.127 250	−8 872	138
70	2.504 550	46 181	2 907	70	1.118 378	−8 734	153
71	2.550 731	49 088	3 230	71	1.109 643	−8 581	169
72	2.599 820	52 318	3 611	72	1.101 062	−8 412	187
73	2.652 138	55 930	4 066	73	1.092 650	−8 225	205
74	2.708 068	59 996	4 614	74	1.084 425	−8 020	224
75	2.768 063	64 609	5 283	75	1.076 405	−7 796	245
76	2.832 673	69 892	6 112	76	1.068 610	−7 550	268
77	2.902 565	76 004	7 156	77	1.061 059	−7 282	292
78	2.978 569	83 160	8 497	78	1.053 777	−6 990	318
79	3.061 729	91 657	10 261	79	1.046 786	−6 672	347
80	3.153 385	101 918	12 647	80	1.040 114	−6 325	379
81	3.255 303	114 565	15 989	81	1.033 789	−5 946	415
82	3.369 868	130 554	20 879	82	1.027 844	−5 531	455
83	3.500 422	151 433	28 453	83	1.022 313	−5 076	502
84	3.651 856	179 886	41 130	84	1.017 237	−4 573	558
85	3.831 742	221 016	64 880	85	1.012 664	−4 016	626
86	4.052 758	285 896	118 167	86	1.008 648	−3 389	715
87	4.338 654	404 063	288 129	87	1.005 259	−2 675	842
88	4.742 717	692 193		88	1.002 584	−1 832	1081
89	5.434 910			89	1.000 752	− 752	
90				90	1.000 000		

TABLE XIII

Values of log F and log E

(See Note, p. 213)

γ	Log F	Δ_1	Δ_2	Log E	Δ_1	Δ_2
45°0	0.2681 2722	3 4688	105	0.1305 4086	2 8279	52
45.1	0.2684 7411	3 4793	105	0.1302 5807	2 8331	52
45.2	0.2688 2204	3 4898	105	0.1299 7476	2 8383	52
45.3	0.2691 7102	3 5004	106	0.1296 9094	2 8434	52
45.4	0.2695 2106	3 5110	106	0.1294 0659	2 8486	51
45.5	0.2698 7216	3 5216	106	0.1291 2174	2 8537	51
45.6	0.2702 2431	3 5322	106	0.1288 3636	2 8589	51
45.7	0.2705 7753	3 5428	107	0.1285 5048	2 8640	51
45.8	0.2709 3181	3 5535	107	0.1282 6408	2 8691	51
45.9	0.2712 8716	3 5642	107	0.1279 7717	2 8742	51
46.0	0.2716 4358	3 5749	108	0.1276 8975	2 8793	51
46.1	0.2720 0108	3 5857	108	0.1274 0182	2 8844	51
46.2	0.2723 5965	3 5965	108	0.1271 1338	2 8894	50
46.3	0.2727 1930	3 6073	108	0.1268 2444	2 8945	50
46.4	0.2730 8003	3 6181	109	0.1265 3499	2 8995	50
46.5	0.2734 4184	3 6290	109	0.1262 4504	2 9045	50
46.6	0.2738 0474	3 6399	109	0.1259 5459	2 9095	50
46.7	0.2741 6873	3 6508	110	0.1256 6364	2 9145	50
46.8	0.2745 3381	3 6618	110	0.1253 7218	2 9195	50
46.9	0.2748 9999	3 6728	110	0.1250 8023	2 9245	50
47.0	0.2752 6727	3 6838	110	0.1247 8778	2 9295	49
47.1	0.2756 3565	3 6948	111	0.1244 9483	2 9344	49
47.2	0.2760 0513	3 7059	111	0.1242 0139	2 9393	49
47.3	0.2763 7572	3 7170	111	0.1239 0746	2 9443	49
47.4	0.2767 4741	3 7281	112	0.1236 1303	2 9492	49
47.5	0.2771 2023	3 7393	112	0.1233 1811	2 9541	49
47.6	0.2774 9415	3 7505	112	0.1230 2271	2 9589	49
47.7	0.2778 6920	3 7617	112	0.1227 2681	2 9638	49
47.8	0.2782 4537	3 7729	113	0.1224 3043	2 9687	48
47.9	0.2786 2266	3 7842	113	0.1221 3357	2 9735	48
48.0	0.2790 0109	3 7955	113	0.1218 3622	2 9783	48
48.1	0.2793 8064	3 8069	114	0.1215 3838	2 9831	48
48.2	0.2797 6133	3 8183	114	0.1212 4007	2 9879	48
48.3	0.2801 4315	3 8297	114	0.1209 4128	2 9927	48
48.4	0.2805 2612	3 8411	115	0.1206 4201	2 9975	48
48.5	0.2809 1023	3 8526	115	0.1203 4226	3 0022	47
48.6	0.2812 9548	3 8641	115	0.1200 4204	3 0070	47
48.7	0.2816 8189	3 8756	116	0.1197 4134	3 0117	47
48.8	0.2820 6945	3 8872	116	0.1194 4017	3 0164	47
48.9	0.2824 5817	3 8988	116	0.1191 3854	3 0211	47
49.0	0.2828 4805	3 9104	117	0.1188 3643	3 0258	47
49.1	0.2832 3909	3 9221	117	0.1185 3385	3 0304	46
49.2	0.2836 3130	3 9338	117	0.1182 3081	3 0351	46
49.3	0.2840 2467	3 9455	118	0.1179 2730	3 0397	46
49.4	0.2844 1923	3 9573	118	0.1176 2333	3 0443	46
49.5	0.2848 1495	3 9691	118	0.1173 1890	3 0489	46
49.6	0.2852 1186	3 9809	119	0.1170 1401	3 0535	46
49.7	0.2856 0996	3 9928	119	0.1167 0866	3 0581	46
49.8	0.2860 0924	4 0047	119	0.1164 0286	3 0626	45
49.9	0.2864 0971	4 0167	120	0.1160 9660	3 0671	45
50.0	0.2868 1137	4 0286	120	0.1157 8988	3 0717	45

TABLE XIII—Continued

γ	Log F	Δ_1	Δ_2	Log E	Δ_1	Δ_2
50°0	0.2868 1137	4 0286	120	0.1157 8988	3 0717	45
50.1	0.2872 1424	4 0406	121	0.1154 8271	3 0762	45
50.2	0.2876 1830	4 0527	121	0.1151 7510	3 0807	45
50.3	0.2880 2357	4 0648	121	0.1148 6703	3 0851	45
50.4	0.2884 3005	4 0769	122	0.1145 5852	3 0896	44
50.5	0.2888 3774	4 0891	122	0.1142 4956	3 0940	44
50.6	0.2892 4665	4 1013	122	0.1139 4016	3 0985	44
50.7	0.2896 5677	4 1135	123	0.1136 3032	3 1028	44
50.8	0.2900 6812	4 1258	123	0.1133 2003	3 1072	44
50.9	0.2904 8070	4 1381	123	0.1130 0931	3 1116	43
51.0	0.2908 9451	4 1504	124	0.1126 9815	3 1159	43
51.1	0.2913 0955	4 1628	124	0.1123 8656	3 1203	43
51.2	0.2917 2584	4 1753	125	0.1120 7453	3 1246	43
51.3	0.2921 4336	4 1877	125	0.1117 6207	3 1289	43
51.4	0.2925 6214	4 2002	125	0.1114 4919	3 1332	43
51.5	0.2929 8216	4 2128	126	0.1111 3587	3 1374	42
51.6	0.2934 0344	4 2254	126	0.1108 2213	3 1417	42
51.7	0.2938 2597	4 2380	127	0.1105 0796	3 1459	42
51.8	0.2942 4977	4 2506	127	0.1101 9337	3 1501	42
51.9	0.2946 7483	4 2634	127	0.1098 7836	3 1543	42
52.0	0.2951 0117	4 2761	128	0.1095 6294	3 1584	41
52.1	0.2955 2878	4 2889	128	0.1092 4709	3 1626	41
52.2	0.2959 5767	4 3017	129	0.1089 3083	3 1667	41
52.3	0.2963 8784	4 3146	129	0.1086 1416	3 1708	41
52.4	0.2968 1930	4 3275	130	0.1082 9707	3 1749	41
52.5	0.2972 5205	4 3405	130	0.1079 7958	3 1790	41
52.6	0.2976 8610	4 3535	130	0.1076 6168	3 1831	40
52.7	0.2981 2144	4 3665	131	0.1073 4338	3 1871	40
52.8	0.2985 5810	4 3796	131	0.1070 2467	3 1911	40
52.9	0.2989 9606	4 3927	132	0.1067 0556	3 1951	40
53.0	0.2994 3533	4 4059	132	0.1063 8605	3 1991	40
53.1	0.2998 7592	4 4191	133	0.1060 6614	3 2030	39
53.2	0.3003 1783	4 4324	133	0.1057 4584	3 2070	39
53.3	0.3007 6107	4 4457	134	0.1054 2514	3 2109	39
53.4	0.3012 0564	4 4591	134	0.1051 0406	3 2148	39
53.5	0.3016 5155	4 4725	134	0.1047 8258	3 2186	38
53.6	0.3020 9880	4 4859	135	0.1044 6072	3 2225	38
53.7	0.3025 4739	4 4994	135	0 1041 3847	3 2263	38
53.8	0.3029 9733	4 5130	136	0.1038 1584	3 2301	38
53.9	0.3034 4863	4 5265	136	0.1034 9283	3 2339	38
54.0	0.3039 0128	4 5402	137	0.1031 6944	3 2377	37
54.1	0.3043 5530	4 5539	137	0.1028 4567	3 2414	37
54.2	0.3048 1069	4 5676	138	0.1025 2153	3 2451	37
54.3	0.3052 6745	4 5814	138	0.1021 9702	3 2488	37
54.4	0.3057 2559	4 5952	139	0.1018 7214	3 2525	37
54.5	0.3061 8511	4 6091	139	0.1015 4689	3 2562	36
54.6	0.3066 4602	4 6230	140	0.1012 2127	3 2598	36
54.7	0.3071 0833	4 6370	140	0.1008 9529	3 2634	36
54.8	0.3075 7203	4 6511	141	0.1005 6895	3 2670	36
54.9	0.3080 3714	4 6652	141	0.1002 4226	3 2705	35
55.0	0.3085 0365	4 6793	142	0.0999 1520	3 2741	35

TABLE XIII—Continued.

γ	Log F	Δ1	Δ2	Log E	Δ1	Δ2
55°0	0.3085 0365	4 6793	142	0.0999 1520	3 2741	35
55.1	0.3089 7158	4 6935	142	0.0995 8779	3 2776	35
55.2	0.3094 4093	4 7077	143	0.0992 6003	3 2811	35
55.3	0.3099 1170	4 7220	143	0.0989 3193	3 2846	34
55.4	0.3103 8391	4 7364	144	0.0986 0347	3 2880	34
55.5	0.3108 5754	4 7508	145	0.0982 7467	3 2914	34
55.6	0.3113 3262	4 7652	145	0.0979 4553	3 2948	34
55.7	0.3118 0915	4 7798	146	0.0976 1605	3 2982	33
55.8	0.3122 8712	4 7943	146	0.0972 8623	3 3015	33
55.9	0.3127 6655	4 8089	147	0.0969 5607	3 3049	33
56.0	0.3132 4745	4 8236	147	0.0966 2559	3 3082	33
56.1	0.3137 2981	4 8384	148	0.0962 9477	3 3114	32
56.2	0.3142 1365	4 8532	149	0.0959 6363	3 3147	32
56.3	0.3146 9896	4 8680	149	0.0956 3216	3 3179	32
56.4	0.3151 8577	4 8829	150	0.0953 0037	3 3211	32
56.5	0.3156 7406	4 8979	150	0.0949 6826	3 3243	31
56.6	0.3161 6385	4 9129	151	0.0946 3583	3 3274	31
56.7	0.3166 5514	4 9280	151	0.0943 0309	3 3305	31
56.8	0.3171 4794	4 9432	152	0.0939 7003	3 3336	31
56.9	0.3176 4226	4 9584	153	0.0936 3667	3 3367	30
57.0	0.3181 3809	4 9736	153	0.0933 0300	3 3397	30
57.1	0.3186 3545	4 9890	154	0.0929 6903	3 3428	30
57.2	0.3191 3435	5 0044	155	0.0926 3475	3 3457	30
57.3	0.3196 3479	5 0198	155	0.0923 0018	3 3487	29
57.4	0.3201 3677	5 0353	156	0.0919 6531	3 3516	29
57.5	0.3206 4030	5 0509	156	0.0916 3014	3 3545	29
57.6	0.3211 4539	5 0666	157	0.0912 9469	3 3574	28
57.7	0.3216 5204	5 0823	158	0.0909 5895	3 3603	28
57.8	0.3221 6027	5 0980	158	0.0906 2292	3 3631	28
57.9	0.3226 7008	5 1139	159	0.0902 8662	3 3659	28
58.0	0.3231 8146	5 1298	160	0.0899 5003	3 3686	27
58.1	0.3236 9444	5 1458	160	0.0896 1317	3 3714	27
58.2	0.3242 0902	5 1618	161	0.0892 7603	3 3741	27
58.3	0.3247 2520	5 1779	162	0.0889 3862	3 3767	26
58.4	0.3252 4299	5 1941	162	0.0886 0095	3 3794	26
58.5	0.3257 6240	5 2104	163	0.0882 6301	3 3820	26
58.6	0.3262 8344	5 2267	164	0.0879 2481	3 3846	26
58.7	0.3268 0611	5 2431	165	0.0875 8635	3 3871	25
58.8	0.3273 3041	5 2595	165	0.0872 4764	3 3897	25
58.9	0.3278 5637	5 2761	166	0.0869 0867	3 3922	25
59.0	0.3283 8397	5 2927	167	0.0865 6945	3 3946	24
59.1	0.3289 1324	5 3094	168	0.0862 2999	3 3971	24
59.2	0.3294 4418	5 3261	168	0.0858 9028	3 3995	24
59.3	0.3299 7679	5 3429	169	0.0855 5033	3 4018	23
59.4	0.3305 1108	5 3598	170	0.0852 1015	3 4042	23
59.5	0.3310 4707	5 3768	171	0.0848 6973	3 4065	23
59.6	0.3315 8475	5 3939	171	0.0845 2908	3 4088	22
59.7	0.3321 2414	5 4110	172	0.0841 8820	3 4110	22
59.8	0.3326 6524	5 4282	173	0.0838 4710	3 4132	22
59.9	0.3332 0806	5 4455	174	0.0835 0578	3 4154	21
60.0	0.3337 5261	5 4629	175	0.0831 6424	3 4176	21

TABLE XIII—Continued

γ	Log F	Δ_1	Δ_2	Log E	Δ_1	Δ_2
60°0	0.3337 5261	5 4629	175	0.0831 6424	3 4176	21
60.1	0.3342 9890	5 4803	175	0.0828 2248	3 4197	21
60.2	0.3348 4694	5 4979	176	0.0824 8051	3 4217	20
60.3	0.3353 9673	5 5155	177	0.0821 3834	3 4238	20
60.4	0.3359 4827	5 5332	178	0.0817 9596	3 4258	20
60.5	0.3365 0159	5 5510	179	0.0814 5338	3 4278	19
60.6	0.3370 5669	5 5688	179	0.0811 1060	3 4297	19
60.7	0.3376 1357	5 5868	180	0.0807 6763	3 4316	19
60.8	0.3381 7225	5 6048	181	0.0804 2446	3 4335	18
60.9	0.3387 3274	5 6229	182	0.0800 8111	3 4354	18
61.0	0.3392 9503	5 6412	183	0.0797 3758	3 4372	18
61.1	0.3398 5915	5 6595	184	0.0793 9386	3 4389	17
61.2	0.3404 2509	5 6778	185	0.0790 4997	3 4407	17
61.3	0.3409 9288	5 6963	186	0.0787 0590	3 4424	17
61.4	0.3415 6251	5 7149	187	0.0783 6167	3 4440	16
61.5	0.3421 3400	5 7336	188	0.0780 1727	3 4456	16
61.6	0.3427 0735	5 7523	188	0.0776 7270	3 4472	15
61.7	0.3432 8258	5 7712	189	0.0773 2798	3 4488	15
61.8	0.3438 5970	5 7901	190	0.0769 8310	3 4503	15
61.9	0.3444 3871	5 8091	191	0.0766 3807	3 4518	14
62.0	0.3450 1962	5 8283	192	0.0762 9290	3 4532	14
62.1	0.3456 0245	5 8475	193	0.0759 4758	3 4546	14
62.2	0.3461 8720	5 8668	194	0.0756 0212	3 4560	13
62.3	0.3467 7388	5 8863	195	0.0752 5652	3 4573	13
62.4	0.3473 6250	5 9058	196	1.0749 1079	3 4586	12
62.5	0.3479 5308	5 9254	197	0.0745 6494	3 4598	12
62.6	0.3485 4562	5 9451	198	0.0742 1895	3 4610	12
62.7	0.3491 4014	5 9650	199	0.0738 7285	3 4622	11
62.8	0.3497 3664	5 9849	200	0.0735 2664	3 4633	11
62.9	0.3503 3513	6 0050	202	0.0731 8030	3 4644	10
63.0	0.3509 3563	6 0251	203	0.0728 3387	3 4654	10
63.1	0.3515 3814	6 0454	204	0.0724 8732	3 4664	10
63.2	0.3521 4268	6 0658	205	0.0721 4068	3 4674	9
63.3	0.3527 4925	6 0862	206	0.0717 9394	3 4683	9
63.4	0.3533 5787	6 1068	207	0.0714 4711	3 4692	8
63.5	0.3539 6856	6 1275	208	0.0711 0019	3 4700	8
63.6	0.3545 8131	6 1483	209	0.0707 5319	3 4708	8
63.7	0.3551 9614	6 1693	210	0.0704 0610	3 4716	7
63.8	0.3558 1307	6 1903	212	0.0700 5895	3 4723	7
63.9	0.3564 3211	6 2115	213	0.0697 1172	3 4729	6
64.0	0.3570 5325	6 2328	214	0.0693 6442	3 4736	6
64.1	0.3576 7653	6 2542	215	0.0690 1706	3 4741	5
64.2	0.3583 0195	6 2757	216	0.0686 6965	3 4747	5
64.3	0.3589 2952	6 2974	218	0.0683 2218	3 4752	4
64.4	0.3595 5926	6 3191	219	0.0679 7466	3 4756	4
64.5	0.3601 9117	6 3410	220	0.0676 2710	3 4760	4
64.6	0.3608 2527	6 3630	221	0.0672 7950	3 4764	3
64.7	0.3614 6158	6 3852	223	0.0669 3186	3 4767	3
64.8	0.3621 0009	6 4075	224	0.0665 8420	3 4769	2
64.9	0.3627 4084	6 4299	225	0.0662 3650	3 4772	2
65.0	0.3633 8383	6 4524	227	0.0658 8379	3 4773	1

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TABLE XIII—Continued

γ	Log F	Δ_1	Δ_2	Log E	Δ_1	Δ_2
65°0	0.3633 8383	6 4524	227	0.0658 8879	3 4773	1
65.1	0.3640 2907	6 4751	228	0.0655 4106	3 4774	1
65.2	0.3646 7658	6 4979	229	0.0651 9331	3 4775	+0
65.3	0.3653 2637	6 5209	231	0.0648 4556	3 4775	−0
65.4	0.3659 7846	6 5439	232	0.0644 9781	3 4775	1
65.5	0.3666 3286	6 5672	234	0.0641 5005	3 4775	1
65.6	0.3672 8957	6 5905	235	0.0638 0231	3 4773	2
65.7	0.3679 4863	6 6141	237	0.0634 5457	3 4772	2
65.8	0.3686 1003	6 6377	238	0.0631 0686	3 4769	3
65.9	0.3692 7380	6 6615	239	0.0627 5916	3 4767	3
66.0	0.3699 3995	6 6855	241	0.0624 1150	3 4764	4
66.1	0.3706 0850	6 7096	242	0.0620 6386	3 4760	4
66.2	0.3712 7946	6 7338	244	0.0617 1626	3 4756	5
66.3	0.3719 5284	6 7582	246	0.0613 6870	3 4751	5
66.4	0.3726 2866	6 7828	247	0.0610 2119	3 4746	6
66.5	0.3733 0694	6 8075	249	0.0606 7373	3 4740	6
66.6	0.3739 8768	6 8324	250	0.0603 2633	3 4734	7
66.7	0.3746 7092	6 8574	252	0.0599 7899	3 4727	7
66.8	0.3753 5666	6 8826	254	0.0596 3172	3 4720	8
66.9	0.3760 4492	6 9080	255	0.0592 8453	3 4712	8
67.0	0.3767 3572	6 9335	257	0.0589 3741	3 4703	9
67.1	0.3774 2907	6 9592	259	0.0585 9037	3 4695	9
67.2	0.3781 2499	6 9851	260	0.0582 4343	3 4685	10
67.3	0.3788 2349	7 0111	262	0.0578 9658	3 4675	11
67.4	0.3795 2460	7 0373	264	0.0575 4983	3 4664	11
67.5	0.3802 2833	7 0637	266	0.0572 0318	3 4653	12
67.6	0.3809 3471	7 0903	268	0.0568 5665	3 4642	12
67.7	0.3816 4373	7 1170	269	0.0565 1023	3 4629	13
67.8	0.3823 5544	7 1440	271	0.0561 6394	3 4617	13
67.9	0.3830 6984	7 1711	273	0.0558 1777	3 4603	14
68.0	0.3837 8695	7 1984	275	0.0554 7174	3 4589	15
68.1	0.3845 0679	7 2259	277	0.0551 2585	3 4575	15
68.2	0.3852 2938	7 2536	279	0.0547 8011	3 4559	16
68.3	0.3859 5475	7 2815	281	0.0544 3451	3 4544	16
68.4	0.3866 8290	7 3096	283	0.0540 8908	3 4527	17
68.5	0.3874 1386	7 3379	285	0.0537 4380	3 4510	18
68.6	0.3881 4765	7 3664	287	0.0533 9870	3 4493	18
68.7	0.3888 8429	7 3951	289	0.0530 5377	3 4475	19
68.8	0.3896 2380	7 4240	291	0.0527 0903	3 4456	19
68.9	0.3903 6620	7 4531	293	0.0523 6447	3 4436	20
69.0	0.3911 1152	7 4825	296	0.0520 2010	3 4416	21
69.1	0.3918 5977	7 5120	298	0.0516 7594	3 4396	21
69.2	0.3926 1097	7 5418	300	0.0513 3198	3 4375	22
69.3	0.3933 6515	7 5718	302	0.0509 8824	3 4353	23
69.4	0.3941 2234	7 6020	305	0.0506 4471	3 4330	23
69.5	0.3948 8254	7 6325	307	0.0503 0141	3 4307	24
69.6	0.3956 4579	7 6632	309	0.0499 5834	3 4283	24
69.7	0.3964 1211	7 6941	312	0.0496 1551	3 4259	25
69.8	0.3971 8152	7 7253	314	0.0492 7292	3 4233	26
69.9	0.3979 5405	7 7567	317	0.0489 3059	3 4208	26
70.0	0.3987 2972	7 7883	319	0.0485 8851	3 4181	27

TABLE XIII—Continued

γ	Log F	Δ_1	Δ_2	Log E	Δ_1	Δ_2
70°0	0.3987 2972	7 7883	319	0.0485 8851	3 4181	27
70.1	0.3995 0855	7 8202	322	0.0482 4670	3 4154	28
70.2	0.4002 9058	7 8524	324	0.0479 0516	3 4126	29
70.3	0.4010 7582	7 8848	327	0.0475 6390	3 4098	29
70.4	0.4018 6430	7 9175	329	0.0472 2292	3 4068	30
70.5	0.4026 5605	7 9504	332	0.0468 8224	3 4039	31
70.6	0.4034 5109	7 9836	335	0.0465 4185	3 4008	31
70.7	0.4042 4945	8 0171	337	0.0462 0177	3 3977	32
70.8	0.4050 5116	8 0508	340	0.0458 6201	3 3945	33
70.9	0.4058 5625	8 0849	343	0.0455 2256	3 3912	33
71.0	0.4066 6474	8 1192	346	0.0451 8344	3 3879	34
71.1	0.4074 7666	8 1538	349	0.0448 4465	3 3844	35
71.2	0.4082 9204	8 1887	352	0.0445 0621	3 3810	36
71.3	0.4091 1090	8 2239	355	0.0441 6812	3 3774	36
71.4	0.4099 3329	8 2594	358	0.0438 3038	3 3738	37
71.5	0.4107 5923	8 2952	361	0.0434 9300	3 3700	38
71.6	0.4115 8875	8 3313	364	0.0431 5600	3 3663	39
71.7	0.4124 2187	8 3677	367	0.0428 1937	3 3624	39
71.8	0.4132 5864	8 4044	371	0.0424 8313	3 3585	40
71.9	0.4140 9909	8 4415	374	0.0421 4729	3 3544	41
72.0	0.4149 4324	8 4789	377	0.0418 1184	3 3504	42
72.1	0.4157 9112	8 5166	381	0.0414 7681	3 3462	42
72.2	0.4166 4279	8 5547	384	0.0411 4219	3 3419	43
72.3	0.4174 9826	8 5931	388	0.0408 0799	3 3376	44
72.4	0.4183 5757	8 6319	391	0.0404 7423	3 3332	45
72.5	0.4192 2076	8 6710	395	0.0401 4091	3 3287	46
72.6	0.4200 8786	8 7105	399	0.0398 0804	3 3241	46
72.7	0.4209 5891	8 7503	402	0.0394 7563	3 3195	47
72.8	0.4218 3394	8 7906	406	0.0391 4368	3 3148	48
72.9	0.4227 1300	8 8312	410	0.0388 1220	3 3099	49
73.0	0.4235 9612	8 8722	414	0.0384 8121	3 3050	50
73.1	0.4244 8334	8 9136	418	0.0381 5070	3 3001	51
73.2	0.4253 7470	8 9554	422	0.0378 2070	3 2950	52
73.3	0.4262 7023	8 9976	426	0.0374 9120	3 2898	52
73.4	0.4271 6999	9 0402	430	0.0371 6221	3 2846	53
73.5	0.4280 7401	9 0832	435	0.0368 3375	3 2793	54
73.6	0.4289 8233	9 1267	439	0.0365 0582	3 2739	55
73.7	0.4298 9499	9 1706	443	0.0361 7843	3 2684	56
73.8	0.4308 1205	9 2149	448	0.0358 5160	3 2628	57
73.9	0.4317 3354	9 2597	452	0.0355 2532	3 2571	58
74.0	0.4326 5950	9 3049	457	0.0351 9961	3 2513	59
74.1	0.4335 9000	9 3506	462	0.0348 7448	3 2455	60
74.2	0.4345 2506	9 3968	467	0.0345 4993	3 2395	60
74.3	0.4354 6474	9 4435	472	0.0342 2598	3 2335	61
74.4	0.4364 0909	9 4906	477	0.0339 0263	3 2273	62
74.5	0.4373 5815	9 5583	482	0.0335 7989	3 2211	63
74.6	0.4383 1198	9 5865	487	0.0332 5778	3 2148	64
74.7	0.4392 7063	9 6352	492	0.0329 3630	3 2084	65
74.8	0.4402 3414	9 6844	498	0.0326 1546	3 2019	66
74.9	0.4412 0258	9 7341	503	0.0322 9528	3 1952	67
75.0	0.4421 7599	9 7844	509	0.0319 7575	3 1885	68

TABLE XIII—Continued

γ	Log F	Δ_1	Δ_2	Log E	Δ_1	Δ_2
75°0	0.4421 7599	9 7344	509	0.0319 7575	3 1885	68
75.1	0.4431 5444	9 8353	514	0.0316 5690	3 1817	69
75.2	0.4441 3797	9 8867	520	0.0313 3872	3 1748	70
75.3	0.4451 2664	9 9387	526	0.0310 2124	3 1678	71
75.4	0.4461 2051	9 9913	532	0.0307 0446	3 1607	72
75.5	0.4471 1965	10 0446	538	0.0303 8839	3 1535	73
75.6	0.4481 2410	10 0984	544	0.0300 7304	3 1462	74
75.7	0.4491 3394	10 1528	551	0.0297 5842	3 1388	75
75.8	0.4501 4922	10 2079	557	0.0294 4454	3 1313	76
75.9	0.4511 7001	10 2637	564	0.0291 3141	3 1237	77
76.0	0.4521 9638	10 3201	571	0.0288 1904	3 1159	78
76.1	0.4532 2839	10 3771	578	0.0285 0745	3 1081	79
76.2	0.4542 6610	10 4349	585	0.0281 9664	3 1002	80
76.3	0.4553 0959	10 4934	592	0.0278 8663	3 0921	82
76.4	0.4563 5893	10 5526	599	0.0275 7742	3 0839	83
76.5	0.4574 1419	10 6126	607	0.0272 6902	3 0757	84
76.6	0.4584 7545	10 6733	615	0.0269 6145	3 0673	85
76.7	0.4595 4278	10 7347	622	0.0266 5472	3 0588	86
76.8	0.4606 1625	10 7970	630	0.0263 4884	3 0502	87
76.9	0.4616 9594	10 8600	639	0.0260 4382	3 0415	88
77.0	0.4627 8195	10 9239	647	0.0257 3967	3 0327	89
77.1	0.4638 7433	10 9886	656	0.0254 3640	3 0237	91
77.2	0.4649 7319	11 0541	664	0.0251 3403	3 0147	92
77.3	0.4660 7860	11 1206	673	0.0248 3257	3 0055	93
77.4	0.4671 9066	11 1879	682	0.0245 3202	2 9962	94
77.5	0.4683 0945	11 2561	692	0.0242 3240	2 9868	95
77.6	0.4694 3506	11 3253	701	0.0239 3372	2 9772	97
77.7	0.4705 6760	11 3954	711	0.0236 3600	2 9676	98
77.8	0.4717 0714	11 4665	721	0.0233 3925	2 9578	99
77.9	0.4728 5379	11 5386	731	0.0230 4347	2 9479	100
78.0	0.4740 0766	11 6118	742	0.0227 4868	2 9378	102
78.1	0.4751 6884	11 6860	753	0.0224 5490	2 9277	103
78.2	0.4763 3743	11 7612	764	0.0221 6213	2 9174	104
78.3	0.4775 1355	11 8376	775	0.0218 7039	2 9070	105
78.4	0.4786 9731	11 9150	786	0.0215 7969	2 8964	107
78.5	0.4798 8881	11 9937	798	0.0212 9005	2 8858	108
78.6	0.4810 8818	12 0735	810	0.0210 0148	2 8750	109
78.7	0.4822 9553	12 1545	823	0.0207 1398	2 8640	111
78.8	0.4835 1098	12 2368	835	0.0204 2758	2 8529	112
78.9	0.4847 3466	12 3203	848	0.0201 4229	2 8417	113
79.0	0.4859 6669	12 4052	862	0.0198 5811	2 8304	115
79.1	0.4872 0721	12 4914	876	0.0195 7507	2 8189	116
79.2	0.4884 5635	12 5789	890	0.0192 9318	2 8073	118
79.3	0.4897 1424	12 6679	904	0.0190 1246	2 7955	119
79.4	0.4909 8103	12 7583	919	0.0187 3291	2 7836	120
79.5	0.4922 5687	12 8503	934	0.0184 5454	2 7716	122
79.6	0.4935 4189	12 9437	950	0.0181 7739	2 7594	123
79.7	0.4948 3626	13 0387	966	0.0179 0145	2 7470	125
79.8	0.4961 4013	13 1353	983	0.0176 2675	2 7345	126
79.9	0.4974 5367	13 2336	1000	0.0173 5330	2 7219	128
80.0	0.4987 7703	13 3336	1018	0.0170 8111	2 7091	129

TABLE XIII—Continued

γ	Log F	Δ_1	Δ_2	Log E	Δ_1	Δ_2
80°0	0.4987 7703	13 3336	1018	0.0170 8111	2 7091	129
80.1	0.5001 1040	13 4354	1036	0.0168 1020	2 6962	131
80.2	0.5014 5394	13 5390	1054	0.0165 4058	2 6831	132
80.3	0.5028 0783	13 6444	1073	0.0162 7227	2 6698	134
80.4	0.5041 7227	13 7517	1093	0.0160 0529	2 6564	136
80.5	0.5055 4744	13 8610	1113	0.0157 3965	2 6429	137
80.6	0.5069 3354	13 9724	1134	0.0154 7536	2 6291	139
80.7	0.5083 3078	14 0858	1156	0.0152 1245	2 6153	140
80.8	0.5097 3936	14 2014	1178	0.0149 5092	2 6012	142
80.9	0.5111 5949	14 3192	1201	0.0146 9080	2 5870	144
81.0	0.5125 9141	14 4393	1225	0.0144 3210	2 5726	145
81.1	0.5140 3534	14 5617	1249	0.0141 7484	2 5581	147
81.2	0.5154 9151	14 6867	1274	0.0139 1903	2 5433	149
81.3	0.5169 6018	14 8141	1300	0.0136 6470	2 5285	151
81.4	0.5184 4159	14 9441	1327	0.0134 1185	2 5134	152
81.5	0.5199 3600	15 0769	1355	0.0131 6052	2 4981	154
81.6	0.5214 4369	15 2124	1384	0.0129 1070	2 4827	156
81.7	0.5229 6493	15 3508	1414	0.0126 6243	2 4671	158
81.8	0.5245 0001	15 4922	1445	0.0124 1572	2 4513	160
81.9	0.5260 4923	15 6366	1477	0.0121 7058	2 4354	162
82.0	0.5276 1289	15 7843	1510	0.0119 2704	2 4192	163
82.1	0.5291 9132	15 9352	1544	0.0116 8512	2 4029	165
82.2	0.5307 8485	16 0896	1579	0.0114 4483	2 3863	167
82.3	0.5323 9381	16 2476	1616	0.0112 0620	2 3696	169
82.4	0.5340 1857	16 4092	1655	0.0109 6924	2 3527	171
82.5	0.5356 5949	16 5747	1694	0.0107 3397	2 3356	173
82.6	0.5373 1696	16 7441	1736	0.0105 0041	2 3183	175
82.7	0.5389 9137	16 9177	1779	0.0102 6859	2 3007	177
82.8	0.5406 8313	17 0955	1823	0.0100 3851	2 2830	179
82.9	0.5423 9268	17 2778	1870	0.0098 1021	2 2651	181
83.0	0.5441 2047	17 4648	1918	0.0095 8371	2 2469	184
83.1	0.5458 6695	17 6566	1968	0.0093 5902	2 2285	186
83.2	0.5476 3260	17 8534	2021	0.0091 3616	2 2100	188
83.3	0.5494 1795	18 0555	2076	0.0089 1517	2 1912	190
83.4	0.5512 2350	18 2631	2133	0.0086 9605	2 1721	193
83.5	0.5530 4980	18 4764	2193	0.0084 7884	2 1529	195
83.6	0.5548 9744	18 6956	2255	0.0082 6355	2 1334	197
83.7	0.5567 6700	18 9211	2320	0.0080 5021	2 1137	199
83.8	0.5586 5912	19 1532	2389	0.0078 3884	2 0937	202
83.9	0.5605 7443	19 3921	2460	0.0076 2947	2 0735	204
84.0	0.5625 1364	19 6381	2535	0.0074 2211	2 0531	207
84.1	0.5644 7745	19 8916	2614	0.0072 1680	2 0324	209
84.2	0.5664 6661	20 1531	2697	0.0070 1356	2 0115	212
84.3	0.5684 8192	20 4228	2784	0.0068 1241	1 9903	214
84.4	0.5705 2420	20 7012	2875	0.0066 1338	1 9689	217
84.5	0.5725 9431	20 9887	2972	0.0064 1649	1 9472	220
84.6	0.5746 9318	21 2859	3073	0.0062 2177	1 9252	222
84.7	0.5768 2177	21 5932	3180	0.0060 2925	1 9029	225
84.8	0.5789 8109	21 9112	3293	0.0058 3896	1 8804	228
84.9	0.5811 7221	22 2405	3413	0.0056 5092	1 8576	231
85.0	0.5833 9626	22 5818	3539	0.0054 6516	1 8345	234

TABLE XIII—Continued

γ	Log F	Δ_1	Δ_2	Log E	Δ_1	Δ_2
85°0	0.5833 9626	22 5818	3539	0.0054 6516	1 8345	234
85.1	0.5856 5444	22 9357	3673	0.0052 8171	1 8111	237
85.2	0.5879 4801	23 3031	3816	0.0051 0060	1 7874	240
85.3	0.5902 7832	23 6846	3967	0.0049 2185	1 7634	243
85.4	0.5926 4679	24 0813	4127	0.0047 4551	1 7391	246
85.5	0.5950 5492	24 4940	4299	0.0045 7160	1 7145	249
85.6	0.5975 0432	24 9239	4481	0.0044 0015	1 6896	253
85.7	0.5999 9671	25 3720	4676	0.0042 3119	1 6643	256
85.8	0.6025 3391	25 8396	4885	0.0040 6476	1 6387	260
85.9	0.6051 1788	26 3281	5109	0.0039 0089	1 6127	263
86.0	0.6077 5069	26 8390	5349	0.0037 3962	1 5864	267
86.1	0.6104 3459	27 3739	5607	0.0035 8097	1 5598	270
86.2	0.6131 7198	27 9346	5886	0.0034 2499	1 5327	274
86.3	0.6159 6543	28 5231	6186	0.0032 7172	1 5053	278
86.4	0.6188 1775	29 1418	6512	0.0031 2118	1 4775	282
86.5	0.6217 3193	29 7929	6865	0.0029 7343	1 4493	286
86.6	0.6247 1122	30 4794	7248	0.0028 2850	1 4207	290
86.7	0.6277 5916	31 2042	7667	0.0026 8642	1 3917	295
86.8	0.6308 7958	31 9709	8124	0.0025 4725	1 3622	299
86.9	0.6340 7668	32 7834	8626	0.0024 1103	1 3323	304
87.0	0.6373 5501	33 6459	9177	0.0022 7779	1 3020	308
87.1	0.6407 1961	34 5636	9785	0.0021 4759	1 2712	313
87.2	0.6441 7597	35 5422	10459	0.0020 2048	1 2398	318
87.3	0.6477 3019	36 5881	11208	0.0018 9649	1 2080	324
87.4	0.6513 8900	37 7089	12043	0.0017 7569	1 1757	329
87.5	0.6551 5989	38 9132	12980	0.0016 5813	1 1428	335
87.6	0.6590 5121	40 2112	14035	0.0015 4385	1 1093	340
87.7	0.6630 7233	41 6147	15230	0.0014 3292	1 0753	347
87.8	0.6672 3380	43 1377	16590	0.0013 2540	1 0406	353
87.9	0.6715 4757	44 7967	18149	0.0012 2134	1 0053	360
88.0	0.6760 2724	46 6116	19948	0.0011 2081	9693	367
88.1	0.6806 8840	48 6064	22040	0.0010 2387	9327	374
88.2	0.6855 4904	50 8104	24492	0.0009 3060	8953	382
88.3	0.6906 3009	53 2597	27396	0.0008 4107	8571	390
88.4	0.6959 5605	55 9993	30870	0.0007 5536	8181	399
88.5	0.7015 5598	59 0862	35077	0.0006 7355	7782	408
88.6	0.7074 6460	62 5940	40245	0.0005 9573	7374	418
88.7	0.7137 2400	66 6184	46693	0.0005 2199	6956	429
88.8	0.7203 8584	71 2878	54895	0.0004 5242	6527	441
88.9	0.7275 1462	76 7773	65561	0.0003 8715	6087	453
89.0	0.7351 9234	83 3334	79812	0.0003 2628	5633	467
89.1	0.7435 2568	91 3146	99496	0.0002 6995	5166	483
89.2	0.7526 5714	101 2642	127847	0.0002 1829	4683	501
89.3	0.7627 8356	114 0489	170975	0.0001 7146	4181	522
89.4	0.7741 8844	131 1464	241655	0.0001 2965	3660	546
89.5	0.7873 0308	155 3119	370693	0.0000 9305	3114	576
89.6	0.8028 3427	192 3813	650756	0.0000 6192	2538	615
89.7	0.8220 7240	257 4569	1501510	0.0000 3654	1923	670
89.8	0.8478 1809	407 6079		0.0000 1731	1253	774
89.9	0.8885 7889			0.0000 0479	479	
90.0	Inf.			0.0000 0000		

The preceding table of logarithms of the elliptic integrals of the first and second kinds is taken from Legendre's *Traité des Fonctions Elliptiques*, volume 2, Table I. The values from 45° to 90° are given for intervals of 0°.1. The values from 0° to 45°, which are comparatively seldom required, have been omitted. For formula and table to be used in interpolation, see page 214.

TABLE XIV

Binominal Coefficients for Interpolation by Differences

k	Coefficients of Δ_2 and Δ_3		k	Coefficients of Δ_2 and Δ_3		k	Coefficients of Δ_2 and Δ_3		k	Coefficients of Δ_2 and Δ_3	
	K_2	K_3		K_2	K_3		K_2	K_3		K_2	K_3
0.01	−0.005	+0.003	0.26	−0.096	+0.056	0.51	−0.125	+0.062	0.76	−0.091	+0.038
.02	− .010	+ .006	.27	− .099	+ .057	.52	− .125	+ .062	.77	− .089	+ .036
.03	− .015	+ .010	.28	− .101	+ .058	.53	− .125	+ .061	.78	− .086	+ .035
.04	− .019	+ .013	.29	− .103	+ .059	.54	− .124	+ .060	.79	− .083	+ .033
.05	− .024	+ .015	.30	− .105	+ .060	.55	− .124	+ .060	.80	− .080	+ .032
.06	− .028	+ .018	.31	− .107	+ .060	.56	− .124	+ .059	.81	− .077	+ .031
.07	− .033	+ .021	.32	− .109	+ .061	.57	− .123	+ .058	.82	− .074	+ .029
.08	− .037	+ .024	.33	− .111	+ .062	.58	− .122	+ .058	.83	− .071	+ .028
.09	− .041	+ .026	.34	− .112	+ .062	.59	− .121	+ .057	.84	− .067	+ .026
.10	− .045	+ .028	.35	− .114	+ .063	.60	− .120	+ .056	.85	− .064	+ .024
.11	− .049	+ .031	.36	− .115	+ .063	.61	− .119	+ .055	.86	− .060	+ .023
.12	− .053	+ .033	.37	− .117	+ .063	.62	− .118	+ .054	.87	− .057	+ .021
.13	− .057	+ .035	.38	− .118	+ .064	.63	− .117	+ .053	.88	− .053	+ .020
.14	− .060	+ .037	.39	− .119	+ .064	.64	− .115	+ .052	.89	− .049	+ .018
.15	− .064	+ .039	.40	− .120	+ .064	.65	− .114	+ .051	.90	− .045	+ .016
.16	− .067	+ .041	.41	− .121	+ .064	.66	− .112	+ .050	.91	− .041	+ .015
.17	− .071	+ .043	.42	− .122	+ .064	.67	− .111	+ .049	.92	− .037	+ .013
.18	− .074	+ .045	.43	− .123	+ .064	.68	− .109	+ .048	.93	− .033	+ .012
.19	− .077	+ .046	.44	− .123	+ .064	.69	− .107	+ .047	.94	− .028	+ .010
.20	− .080	+ .048	.45	− .124	+ .064	.70	− .105	+ .045	.95	− .024	+ .008
.21	− .083	+ .049	.46	− .124	+ .064	.71	− .103	+ .044	.96	− .019	+ .007
.22	− .086	+ .051	.47	− .125	+ .064	.72	− .101	+ .043	.97	− .015	+ .005
.23	− .089	+ .052	.48	− .125	+ .063	.73	− .099	+ .042	.98	− .010	+ .003
.24	− .091	+ .053	.49	− .125	+ .063	.74	− .096	+ .040	.99	− .005	+ .002
.25	− .094	+ .055	.50	− .125	+ .063	.75	− .094	+ .039	1.00	− .000	+ .000

INTERPOLATION FORMULA

$$f(a+h)=f(a)+k\triangle_1+\frac{k(k-1)}{2!}\triangle_2+\frac{k(k-1)(k-2)}{3!}\triangle_3$$
$$+\frac{k(k-1)(k-2)(k-3)}{4!}\triangle_4+\cdots\quad (a)$$

or, $f(a+h)=f(a)+k\triangle_1+K_2\triangle_2+K_3\triangle_3+\cdots \quad (b)$

where the constants K_2 and K_3 are given in the above table as functions of k and

$$k=\frac{h}{\delta}$$

where h is the remainder above the value of a for which the function is given in the table, and δ is the increment of a in the table.

ILLUSTRATION

To find the value of log F for $49^\circ\ 15'\ 36''=49^\circ.260$

For $49^\circ.2$ log $F=0.2836\ 3130=f(a)$

$h=.06$, $\delta=0.1$ $k=0.6$

From Table XIV, $K_2=-.120$

$K_3=+.056$

From Table XIII,

$\triangle_1=39338$

$\triangle_2=\quad 117$

$\triangle_3=\quad 1$

Substituting these values of K_2, K_3, $\triangle_1$, $\triangle_2$, $\triangle_3$ in formula (b) above we have as the value of log F for the given angle

log $F=0.2836\ 3130+0.0002\ 3603-0.0000\ 0014=0.2838\ 6719.$

TABLE XV

Values of the Quantities $q-\frac{l}{2}$ *or* $q_1-\frac{l_1}{2}$ *and* $Log_{10}(1+\epsilon)$ *with Argument* q *or* q_1

$$q-\frac{l}{2}=2\left(\frac{l}{2}\right)^5+15\left(\frac{l}{2}\right)^9+\cdots$$

$$\epsilon=3q^4-4q^6+9q^8-12q^{10}+\cdots$$

$$q_1-\frac{l_1}{2}=2\left(\frac{l_1}{2}\right)^5+15\left(\frac{l_1}{2}\right)^9+\cdots$$

(For use with Formulas (8), (9), (45), (76), (77), and (78))

q or q_1	$q-\frac{l}{2}$ or $q_1-\frac{l_1}{2}$	△	ε	△	$\text{Log}_{10}(1+\varepsilon)$	△
0.020	0.000 00001	0	0.000 00048	22	0.000 00021	9
.022	.000 00001	1	.000 00070	29	.000 00030	13
.024	.000 00002	0	.000 00099	38	.000 00043	16
.026	.000 00002	1	.000 00137	47	.000 00059	21
.028	.000 00003	2	.000 00184	59	.000 00080	25
0.030	0.000 00005	2	0.000 00243	71	0.000 00105	31
.032	.000 00007	2	.000 00314	86	.000 00136	38
.034	.000 00009	3	.000 00400	103	.000 00174	44
.036	.000 00012	4	.000 00503	121	.000 00218	53
.038	.000 00016	5	.000 00624	142	.000 00271	61
0.040	0.000 00021	5	0.000 00766	165	0.000 00332	72
.042	.000 00026	7	.000 00931	191	.000 00404	83
.044	.000 00033	8	.000 01122	217	.000 00487	94
.046	.000 00041	10	.000 01339	249	.000 00581	109
.048	.000 00051	12	.000 01588	280	.000 00690	122
0.050	0.000 00063	13	0.000 01868	318	0.000 00812	138
.052	.000 00076	16	.000 02186	355	.000 00950	154
.054	.000 00092	18	.000 02541	397	.000 01104	172
.056	.000 00110	21	.000 02938	442	.000 01276	192
.058	.000 00131	25	.000 03380	490	.000 01468	213
0.060	0.000 00156	27	0.000 03870	540	0.000 01681	234
.062	.000 00183	32	.000 04410	596	.000 01915	259
.064	.000 00215	36	.000 05006	654	.000 02174	283
.066	.000 00251	40	.000 05660	715	.000 02457	312
.068	.000 00291	45	.000 06375	781	.000 02769	339
0.070	0.000 00336	51	0.000 07156	851	0.000 03108	369
.072	.000 00387	57	.000 08007	924	.000 03477	401
.074	.000 00444	63	.000 08931	1002	.000 03878	436
.076	.000 00507	70	.000 09933	1083	.000 04314	470
.078	.000 00577	78	.000 11016	1169	.000 04784	509

TABLE XV—Continued

q or q_1	$q-\frac{1}{2}$ or $q_1-\frac{l_1}{2}$	△	ε	△	$\log_{10}(1+\varepsilon)$	△
0.080	0.000 00655	86	0.000 12185	1259	0.000 05293	545
.082	.000 00741	95	.000 13444	1354	.000 05838	588
.084	.000 00836	105	.000 14798	1453	.000 06426	631
.086	.000 00941	114	.000 16251	1557	.000 07057	676
.088	.000 01055	126	.000 17808	1666	.000 07733	724
0.090	0.000 01181	137	0.000 19474	1779	0.000 08457	772
.092	.000 01318	150	.000 21253	1899	.000 09229	825
.094	.000 01468	162	.000 23152	2022	.000 10054	878
.096	.000 01630	177	.000 25174	2150	.000 10932	937
.098	.000 01807	193	.000 27324	2285	.000 11869	988
0.100	0.000 02000	102	0.000 29609	1194	0.000 12857	519
.101	.000 02102	106	.000 30803	1230	.000 13376	533
.102	.000 02208	110	.000 32033	1266	.000 13909	550
.103	.000 02318	115	.000 33299	1303	.000 14459	566
.104	.000 02433	119	.000 34602	1340	.000 15025	582
0.105	0.000 02552	123	0.000 35942	1379	0.000 15607	598
.106	.000 02675	129	.000 37321	1410	.000 16205	616
.107	.000 02804	134	.000 38731	1465	.000 16821	632
.108	.000 02938	138	.000 40196	1498	.000 17453	651
.109	.000 03076	144	.000 41694	1539	.000 18104	668
0.110	0.000 03220	149	0.000 43233	1581	0.000 18772	686
.111	.000 03369	154	.000 44814	1624	.000 19458	705
.112	.000 03523	160	.000 46438	1667	.000 20163	724
.113	.000 03683	166	.000 48105	1711	.000 20887	742
.114	.000 03849	172	.000 49816	1756	.000 21629	762
0.115	0.000 04021	178	0.000 51572	1802	0.000 22391	783
.116	.000 04199	184	.000 53374	1848	.000 23174	802
.117	.000 04383	191	.000 55222	1895	.000 23976	823
.118	.000 04574	196	.000 57117	1943	.000 24799	843
.119	.000 04770	204	.000 59060	1992	.000 25642	865
0.120	0.000 04974	210	0.000 61052	2041	0.000 26507	885
.121	.000 05184	218	.000 63093	2091	.000 27392	908
.122	.000 05402	226	.000 65184	2143	.000 28300	930
.123	.000 05628	232	.000 67327	2195	.000 29230	953
.124	.000 05860	240	.000 69522	2247	.000 30183	975
0.125	0.000 06100	248	0.000 71769	2301	0.000 31158	998
.126	.000 06348	255	.000 74070	2355	.000 32156	1022
.127	.000 06603	265	.000 76425	2410	.000 33178	1046
.128	.000 06868	272	.000 78835	2466	.000 34224	1071
.129	.000 07140	280	.000 81301	2523	.000 35295	1094

TABLE XV—Continued

q or q_1	$q-\frac{1}{2}$ or $q_1-\frac{l_1}{2}$	△	ε	△	$\text{Log}_{10}(1+\varepsilon)$	△
0.130	0.000 07420	290	0.000 83824	2581	0.000 36389	1120
.131	.000 07710	299	.000 86405	2639	.000 37509	1145
.132	.000 08009	308	.000 89044	2698	.000 38654	1171
.133	.000 08317	317	.000 91742	2759	.000 39825	1196
.134	.000 08634	327	.000 94501	2820	.000 41021	1224
0.135	0.000 08961	336	0.000 97321	2881	0.000 42245	1251
.136	.000 09297	347	.001 00202	2945	.000 43496	1277
.137	.000 09644	357	.001 03147	3012	.000 44773	1305
.138	.000 10001	367	.001 06155	3073	.000 46078	1333
.139	.000 10368	378	.001 09228	3138	.000 47411	1362
0.140	0.000 10746	389	0.001 12366	3204	0.000 48773	1389
.141	.000 11135	401	.001 15570	3272	.000 50162	1420
.142	.000 11536	411	.001 18842	3339	.000 51582	1448
.143	.000 11947	423	.001 22181	3409	.000 53030	1479
.144	.000 12370	435	.001 25590	3479	.000 54509	1509
0.145	0.000 12805	448	0.001 29069	3549	0.000 56018	1539
.146	.000 13253	459	.001 32618	3621	.000 57557	1571
.147	.000 13712	473	.001 36239	3694	.000 59128	1602
.148	.000 14185	485	.001 39933	3768	.000 60730	1634
.149	.000 14670	498	.001 43701	3842	.000 62364	1666
0.150	0.000 15168		0.001 47543		0.000 64030	

Tables XV and XVI are reproduced from Nagaoka's paper; see footnote, page 12.

TABLE XVI

Values of ϵ_1 and $-\epsilon_1'$ with Argument q_1

$$\epsilon_1 = 32\, q_1^3 - 40\, q_1^4 + 48\, q_1^5 - \cdot \cdot \cdot \cdot$$
$$-\epsilon_1' = 8\, q_1^2 - \epsilon_1.$$

(For use with Formulas (9) and (9a))

q_1	ϵ_1	Δ	$-\epsilon_1'$	Δ
0.0100	0.000 03160	93	0.000 76840	1499
.0099	.000 03067	92	.000 75341	1484
.0098	.000 02975	89	.000 73857	1471
.0097	.000 02886	89	.000 72386	1455
.0096	.000 02797	86	.000 70931	1442
0.0095	0.000 02711	84	0.000 69489	1428
.0094	.000 02627	83	.000 68061	1413
.0093	.000 02544	81	.000 66648	1399
.0092	.000 02463	78	.000 65249	1386
.0091	.000 02385	78	.000 63863	1370
0.0090	0.000 02307	76	0.000 62493	1356
.0089	.000 02231	74	.000 61137	1342
.0088	.000 02157	73	.000 59795	1327
.0087	.000 02084	71	.000 58468	1313
.0086	.000 02013	69	.000 57155	1299
0.0085	0.000 01944	67	0.000 55856	1285
.0084	.000 01877	67	.000 54571	1269
.0083	.000 01810	64	.000 53302	1256
.0082	.000 01746	62	.000 52046	1242
.0081	.000 01684	62	.000 50804	1226
0.0080	0.000 01622	60	0.000 49578	1212
.0079	.000 01562	59	.000 48366	1197
.0078	.000 01503	56	.000 47169	1184
.0077	.000 01447	55	.000 45985	1169
.0076	.000 01392	55	.000 44816	1153
0.0075	0.000 01337	52	0.000 43663	1140
.0074	.000 01285	51	.000 42523	1125
.0073	.000 01234	51	.000 41398	1109
.0072	.000 01183	48	.000 40289	1096
.0071	.000 01135	47	.000 39193	1081
0.0070	0.000 01088	46	0.000 38112	1066
.0069	.000 01042	45	.000 37046	1051
.0068	.000 00997	43	.000 35995	1037
.0067	.000 00954	42	.000 34958	1022
.0066	.000 00912	40	.000 33936	1008
0.0065	0.000 00872	40	0.000 32928	992
.0064	.000 00832	38	.000 31936	978
.0063	.000 00794	37	.000 30958	963
.0062	.000 00757	37	.000 29995	947
.0061	.000 00720	34	.000 29048	934

TABLE XVI—Continued

q_1	ϵ_1	Δ	$-\epsilon_1'$	Δ
0.0060	0.000 00686	34	0.000 28114	918
.0059	.000 00652	33	.000 27196	903
.0058	.000 00619	30	.000 26293	890
.0057	.000 00589	31	.000 25403	873
.0056	.000 00558	30	.000 24530	858
0.0055	0.000 00528	27	0.000 23672	845
.0054	.000 00501	28	.000 22827	828
.0053	.000 00473	26	.000 21999	814
.0052	.000 00447	26	.000 21185	798
.0051	.000 00421	24	.000 20387	784
0.0050	0.000 00397	23	0.000 19603	769
.0049	.000 00374	22	.000 18834	754
.0048	.000 00352	22	.000 18080	738
.0047	.000 00330	21	.000 17342	723
.0046	.000 00309	19	.000 16619	709
0.0045	0.000 00290	18	0.000 15910	694
.0044	.000 00272	19	.000 15216	677
.0043	.000 00253	17	.000 14539	663
.0042	.000 00236	16	.000 13876	648
.0041	.000 00220	16	.000 13228	632
0.0040	0.000 00204	15	0.000 12596	617
.0039	.000 00189	14	.000 11979	602
.0038	.000 00175	14	.000 11377	586
.0037	.000 00161	13	.000 10791	571
.0036	.000 00148	12	.000 10220	556
0.0035	0.000 00136	11	0.000 09664	541
.0034	.000 00125	11	.000 09123	525
.0033	.000 00114	9	.000 08598	511
.0032	.000 00105	10	.000 08087	494
.0031	.000 00095	9	.000 07593	479
0.0030	0.000 00086	8	0.000 07114	464
.0029	.000 00078	8	.000 06650	448
.0028	.000 00070	7	.000 06202	433
.0027	.000 00063	7	.000 05769	417
.0026	.000 00056	6	.000 05352	402
0.0025	0.000 00050	6	0.000 04950	386
.0024	.000 00044	5	.000 04564	371
.0023	.000 00039	5	.000 04193	355
.0022	.000 00034	4	.000 03838	340
.0021	.000 00030	4	.000 03498	324
0.0020	0.000 00026	4	0.000 03174	308
.0019	.000 00022	3	.000 02866	293
.0018	.000 00019	3	.000 02573	277
.0017	.000 00016	3	.000 02296	261
.0016	.000 00013	2	.000 02035	246

TABLE XVI—Continued

q_1	ϵ_1	Δ	$-\epsilon_1'$	Δ
0.0015	0.000 00011	2	0.000 01789	230
.0014	.000 00009	2	.000 01559	214
.0013	.000 00007	1	.000 01345	199
.0012	.000 00006	2	.000 01146	182
.0011	.000 00004	1	.000 00964	167
0.0010	0.000 00003	1	0.000 00797	151
.0009	.000 00002	0	.000 00646	136
.0008	.000 00002	1	.000 00510	119
.0007	.000 00001	0	.000 00391	104
.0006	.000 00001	1	.000 00287	87
0.0005	0.000 00000		0.000 00200	72
.0004	.000 00000		.000 00128	56
.0003	.000 00000		.000 00072	40
.0002	.000 00000		.000 00032	24
.0001	.000 00000		.000 00008	

TABLE XVII

Coefficients of the Hypergeometric Series in Formula (18)

Series	a_1	a_2	a_3
$F\left(\frac{1}{12}, \frac{5}{12}, \frac{1}{2}, \frac{J-1}{J}\right)$	0.069 4444	0.035 5260	0.023 8485
$F\left(-\frac{1}{12}, \frac{7}{12}, \frac{1}{2}, \frac{J-1}{J}\right)$	−0.097 2222	−0.047 0358	−0.031 0523
$F\left(\frac{5}{12}, \frac{13}{12}, \frac{3}{2}, \frac{J-1}{J}\right)$	0.300 9259	0.177 6300	0.126 0562
$F\left(\frac{7}{12}, \frac{11}{12}, \frac{3}{2}, \frac{J-1}{J}\right)$	0.356 4814	0.216 3645	0.155 2615

TABLE XVIII

Showing the Location and Magnitude of the Positive and Negative Maxima and the Positions of the Roots of the Coefficients in Gray's and Searle and Airey's Formulas

(For use in Formulas (40), (43), and (56))

$\frac{x}{A}$	X_2	$\frac{x}{A}$	X_4	$\frac{x}{A}$	X_6	$\frac{x}{A}$	X_8
0	3.0000	0	2.5000	0	2.1875	0	1.9688
0.8660	0	0.531	0	0.3898	0	0.3000	0
∞	−∞	1.118	−3.750	0.6961	− 1.8273	0.5162	− 1.2177
		1.489	0	0.9203	0	0.687	0
		∞	+∞	1.737	+30.69	1.1521	+ 7.364
				2.063	0	1.268	0
				∞	−∞	2.309	−570.97
						2.613	0
						∞	+∞

$\frac{x}{A}$	X_{10}	$\frac{x}{A}$	X_{12}	$\frac{x}{A}$	X_{14}
0	1.8407	0	1.6758	0	1.5710
0.2575	0	0.2193	0	0.1936	0
0.4145	− 0.924	0.3466	− 0.756	0.2992	− 0.6446
0.5520	0	0.4629	0	0.4010	0
0.8006	+ 3.428	0.6475	+ 2.000	0.5460	+ 1.396
0.9386	0	0.7627	0	0.6439	0
1.413	−60.80	1.052	− 18.20	0.8515	− 8.166
1.589	0	1.145	0	0.9559	0
2.862	+18892	1.734	+836.1	1.289	+154.6
3.158	0	1.902	0	1.414	0
∞	−∞	3.406	−963500	2.044	−16993
		3.618	0	2.207	0
		∞	+∞	3.950	+70850000
				4.226	0
				∞	−∞

The function X_{2n} has n roots, between which values it makes oscillations of ever-increasing amplitude, and for values of $\frac{x}{A}$ greater than the largest root the function increases rapidly without limit. The functions L_{2n} have the same form as X_{2n}, $\frac{l}{a}$ being the variable instead of $\frac{x}{A}$.

TABLE XIX

Values of Coefficients in Gray's and Searle and Airey's Formulas

(For use in Formulas (40), (43), and (56))

$\frac{x}{A}$	X_2	X_4	X_6	X_8	X_{10}	X_{12}	X_{14}
0.0	+ 3.000	+ 2.500	+ 2.188	+ 1.969	+ 1.841	+ 1.676	+ 1.571
0.1	2.960	2.400	2.015	1.712	1.494	1.234	+ 1.032
0.2	2.840	2.106	1.521	1.017	+ 0.618	+ 0.216	− 0.073
0.3	2.640	1.632	+ 0.780	+ 0.090	− 0.355	− 0.635	− 0.645
0.4	2.360	1.002	− 0.090	− 0.764	− 0.874	− 0.596	− 0.0093
0.5	2.000	+ 0.250	− 0.938	− 1.203	− 0.526	+ 0.483	+ 1.208
0.6	1.560	− 0.580	− 1.577	− 0.909	+ 0.760	+ 1.793	+ 1.000
0.7	1.040	− 1.438	− 1.814	+ 0.228	2.467	+ 1.662	− 3.175
0.8	+ 0.440	− 2.262	− 1.452	+ 2.207	3.423	− 1.733	− 7.231
0.9	− 0.240	− 2.976	− 0.335	+ 4.606	+ 1.924	− 8.748	− 6.811
1.0	− 1.000	− 3.500	+ 1.688	+ 6.719	− 3.878	− 16.46	+ 10.48
1.1	− 1.840	− 3.750	4.673	7.240			
1.2	− 2.760	− 3.606	8.589	+ 4.509	− 31.72	+ 22.27	+119.6
1.3	− 3.760	− 2.976	13.28	− 3.595			
1.4	− 4.840	− 1.734	18.44	− 19.49	− 60.66		+ 29.1
1.5	− 6.000	+ 0.250	+ 23.56	− 46.24	− 48.96	+765.7	−486.5
1.6	− 7.240	+ 3.114	27.90	− 89.42			
1.7	− 8.560	7.008	30.46	− 137.4	+151.0	+818.1	
1.8	− 9.960	12.09	29.83	− 205.2			
1.9	−11.44	18.53	24.50	− 285.9		+ 21.8	
2.0	−13.00	+ 26.50	+ 12.19	− 375.0	+1591	− 1969	−16740
2.1	−14.64	36.55	− 9.64	− 464.9			
2.2	−16.36	47.80	− 44.09	− 538.3	4059	−14090	−1840
2.3	−18.16	62.54	− 104.9	− 570.9			
2.4	−20.04	77.61	− 166.3	− 535.5			
2.5	−22.00	+ 96.24	− 263.4	− 386.7	10908	−80050	
2.6	−24.04	117.6	− 390.4	− 64.4			+ 658400
2.7	−26.16	142.2	− 559.0	+ 505.9			
2.8	−28.36	169.0	− 801.2	1433	18390	−222400	
2.9	−30.64	201.3	−1039.1	2833			
3.0	−33.00	+ 236.5	−1370.3	+4869	+15797	−509200	19132000
3.25	−38.25	+ 343.1	−2553	+14118			
3.5	−46.00	+ 480.2	−4414	+33030	−146970	−893400	33670000
3.75	−53.25	+ 653.1	−7215	+68410		+265600	
4.0	−61.00	+ 866.5	−11286	+130400	-1.229×10^6	$+6.625\times10^6$	59080000
4.25	−69.25						−16530000
4.5	−77.00	+1440.3	−24956	+399000	-5.683×10^6	$+5.972\times10^7$	-4.172×10^8
5.0	−97.00	+2252.5	−49810	+1038700	-2.007×10^7	$+3.463\times10^8$	-4.855×10^9

This table used in conjunction with the preceding should make it possible to investigate the convergence of Gray's or Searle and Airey's formula in any given case. It will also facilitate calculations by these formulas when $\frac{x}{A}$ has one of the values included in the table. This table gives also the values of the L_{2n} coefficients if $\frac{l}{a}$ be taken as argument in place of $\frac{x}{A}$.

TABLE XX

Nagaoka's Table of Values of the Correction Factor for the Ends K, as a Function of the Angle $\theta = \tan^{-1} \frac{2a}{b}$

(For use in Formula (75))

θ	K	Δ_1	Δ_2	θ	K	Δ_1	Δ_2
0°	1.000 000	− 7370	+ 72	45°	0.688 423	− 7659	− 95
1	0.992 630	− 7298	67	46	.680 764	− 7754	− 102
2	.985 332	− 7231	63	47	.673 010	− 7856	− 108
3	.978 101	− 7168	60	48	.665 154	− 7964	− 115
4	.970 933	− 7109	56	49	.657 190	− 8079	− 120
5	0.963 825	− 7053	+ 52	50	0.649 111	− 8199	− 128
6	.956 771	− 7001	47	51	.640 912	− 8327	− 136
7	.949 770	− 6955	44	52	.632 585	− 8463	− 142
8	.942 815	− 6910	40	53	.624 122	− 8605	− 152
9	.935 906	− 6870	37	54	.615 517	− 8757	− 160
10	0.929 036	− 6833	+ 34	55	0.606 760	− 8917	− 169
11	.922 203	− 6799	30	56	.597 843	− 9086	− 179
12	.915 404	− 6769	27	57	.588 757	− 9265	− 190
13	.908 635	− 6742	24	58	.579 492	− 9455	− 200
14	.901 893	− 6718	19	59	.570 037	− 9655	− 214
15	0.895 175	− 6699	+ 18	60	0.560 382	− 9869	− 226
16	.888 476	− 6681	14	61	.550 513	−10095	− 239
17	.881 795	− 6667	10	62	.540 418	−10334	− 256
18	.875 128	− 6657	8	63	.530 084	−10590	− 270
19	.868 471	− 6649	4	64	.519 494	−10860	− 288
20	0.861 822	− 6645	+ 2	65	0.508 634	−11148	− 308
21	.855 177	− 6643	− 2	66	.497 486	−11456	− 328
22	.848 534	− 6645	− 5	67	.486 030	−11784	− 351
23	.841 889	− 6650	− 9	68	.474 246	−12135	− 376
24	.835 239	− 6659	− 10	69	.462 111	−12511	− 403
25	0.828 580	− 6669	− 16	70	0.449 600	−12914	− 435
26	.821 911	− 6685	− 17	71	.436 686	−13349	− 467
27	.815 226	− 6702	− 21	72	.423 337	−13816	− 506
28	.808 524	− 6723	− 24	72	.409 521	−14322	− 549
29	.801 801	− 6747	− 28	74	.395 199	−14871	− 597
30	0.795 054	− 6775	− 32	75	0.380 328	−15468	− 653
31	.788 279	− 6807	− 34	76	.364 860	−16121	− 717
32	.781 472	− 6841	− 39	77	.348 739	−16838	− 791
33	.774 631	− 6880	− 41	78	.331 901	−17629	− 881
34	.767 751	− 6921	− 46	79	.314 272	−18510	− 985
35	0.760 830	− 6967	− 50	80	0.295 762	−19495	− 1116
36	.753 863	− 7017	− 54	81	.276 267	−20611	− 1275
37	.746 846	− 7071	− 57	82	.255 656	−21886	− 1484
38	.739 775	− 7128	− 61	83	.233 770	−23370	− 1758
39	.732 647	− 7189	− 67	84	.210 400	−25128	− 2144
40	0.725 458	− 7256	− 71	85	0.185 272	−27272	− 2725
41	.718 202	− 7327	− 75	86	.158 000	−29997	− 3707
42	.710 875	− 7402	− 81	87	.128 003	−33704	− 5760
43	.703 473	− 7483	− 84	88	.094 299	−39464	−15371
44	.695 990	− 7567	− 92	89	.054 835	−54835	

55376°—16——15

TABLE XXI

Nagaoka's Table of Values of the End Correction K as Function of the Ratio $\frac{Diameter}{Length}$

For use in Formula (75))

$\frac{\text{Diameter}}{\text{Length}}$	K	Δ_1	Δ_2	$\frac{\text{Diameter}}{\text{Length}}$	K	Δ_1	Δ_2
0.00	1.000 000	−4231	+24	0.45	0.833 723	−3160	+21
.01	.995 769	−4207	26	.46	.830 563	−3139	22
.02	.991 562	−4181	24	.47	.827 424	−3117	21
.03	.987 381	−4157	25	.48	.824 307	−3096	21
.04	.983 224	−4132	25	.49	.821 211	−3075	21
0.05	0.979 092	−4107	+25	0.50	0.818 136	−3054	+21
.06	.974 985	−4082	26	.51	.815 082	−3033	21
.07	.970 903	−4056	24	.52	.812 049	−3012	21
.08	.966 847	−4032	24	.53	.809 037	−2991	20
.09	.962 815	−4008	26	.54	.806 046	−2971	21
0.10	0.958 807	−3982	+25	0.55	0.803 075	−2950	+20
.11	.954 825	−3957	24	.56	.800 125	−2930	20
.12	.950 868	−3933	23	.57	.797 195	−2910	20
.13	.946 935	−3910	26	.58	.794 285	−2890	20
.14	.943 025	−3884	27	.59	.791 395	−2870	20
0.15	0.939 141	−3857	+23	0.60	0.788 525	−2850	+19
.16	.935 284	−3834	23	.61	.785 675	−2831	19
.17	.931 450	−3811	26	.62	.782 844	−2812	20
.18	.927 639	−3785	24	.63	.780 032	−2792	19
.19	.923 854	−3761	24	.64	.777 240	−2773	19
0.20	0.920 093	−3737	+24	0.65	0.774 467	−2754	+19
.21	.916 356	−3713	24	.66	.771 713	−2735	19
.22	.912 643	−3689	25	.67	.768 978	−2716	19
.23	.908 954	−3664	23	.68	.766 262	−2697	18
.24	.905 290	−3641	25	.69	.763 565	−2679	18
0.25	0.901 649	−3616	+23	0.70	0.760 886	−2661	+18
.26	.898 033	−3593	24	.71	.758 225	−2643	19
.27	.894 440	−3569	23	.72	.755 582	−2624	17
.28	.890 871	−3546	24	.73	.752 958	−2607	18
.29	.887 325	−3522	24	.74	.750 351	−2589	18
0.30	0.883 803	−3498	+22	0.75	0.747 762	−2571	+17
.31	.880 305	−3476	24	.76	.745 191	−2554	17
.32	.876 829	−3452	23	.77	.742 637	−2537	18
.33	.873 377	−3429	23	.78	.740 100	−2519	17
.34	.869 948	−3406	22	.79	.737 581	−2502	16
0.35	0.866 542	−3384	+24	0.80	0.735 079	−2486	+19
.36	.863 158	−3360	22	.81	.732 593	−2467	16
.37	.859 799	−3338	23	.82	.730 126	−2451	16
.38	.856 461	−3315	22	.83	.727 675	−2435	16
.39	.853 146	−3293	23	.84	.725 240	−2419	17
0.40	0.849 853	−3270	+22	0.85	0.722 821	−2402	+16
.41	.846 583	−3248	23	.86	.720 419	−2386	16
.42	.843 335	−3225	21	.87	.718 033	−2370	15
.43	.840 110	−3204	21	.88	.715 663	−2355	16
.44	.836 906	−3183	23	.89	.713 308	−2339	17

TABLE XXI—Continued

Diameter/Length	K	Δ_1	Δ_2		Diameter/Length	K	Δ_1	Δ_2	Δ_3
0.90	0.710 969	−2322	+ 14		2.50	0.471 865	−9292	+ 405	
.91	.708 647	−2308	16		2.60	.462 573	−8887	378	
.92	.706 339	−2292	15		2.70	.453 686	−8509	355	
.93	.704 047	−2277	16		2.80	.445 177	−8154	330	
.94	.701 770	−2261	14		2.90	.437 023	−7824	312	
0.95	0.699 509	−2247	+ 15		3.00	0.429 199	−7512	+ 293	
.96	.697 262	−2232	15		3.10	.421 687	−7219	275	
.97	.695 030	−2217	15		3.20	.414 468	−6944	260	
.98	.692 813	−2202	14		3.30	.407 524	−6684	245	
.99	.690 611	−2188	14		3.40	.400 840	−6439	230	
1.00	0.688 423	−10726	+ 344		3.50	0.394 401	−6209	+ 220	
1.05	.677 697	−10382	330		3.60	.388 192	−5989	207	
1.10	.667 315	−10052	316		3.70	.382 203	−5782	195	
1.15	.657 263	−9736	303		3.80	.376 421	−5587	186	
1.20	.647 527	−9433	290		3.90	.370 834	−5401	174	
1.25	0.638 094	−9143	+ 278		4.00	0.365 433	−5227	+ 168	
1.30	.628 951	−8865	266		4.10	.360 206	−5059	161	
1.35	.620 086	−8599	255		4.20	.355 147	−4898	152	
1.40	.611 487	−8343	244		4.30	.350 249	−4746	141	
1.45	.603 144	−8099	236		4.40	.345 503	−4605	138	
1.50	0.595 045	−7863	+ 224		4.50	0.340 898	−4467	+ 134	
1.55	.587 182	−7639	215		4.60	.336 431	−4333	125	
1.60	.579 543	−7424	208		4.70	.332 098	−4208	118	
1.65	.572 119	−7216	198		4.80	.327 890	−4090	115	
1.70	.564 903	−7018	190		4.90	.323 800	−3975	102	
1.75	0.557 885	−6828	+ 184		5.00	0.319 825	−18321	+2227	−397
1.80	.551 057	−6644	176		5.50	.301 504	−16094	1830	−306
1.85	.544 413	−6468	170		6.00	.285 410	−14264	1524	−241
1.90	.537 945	−6298	161		6.50	.271 146	−12740	1283	−193
1.95	.531 647	−6137	154		7.00	.258 406	−11457	1090	−153
2.00	0.525 510	−11809	+ 580		7.50	0.246 949	−10367	+ 937	−127
2.10	.513 701	−11229	539		8.00	.236 582	−9430	810	−104
2.20	.502 472	−10690	499		8.50	.227 152	−8620	706	− 86
2.30	.491 782	−10191	465		9.00	.218 532	−7914	620	
2.40	.481 591	−9726	434		9.50	.210 618	−7294		
					10.00	0.203 324			

In the last part of this table several errors in the fifth and sixth places of decimals have been corrected.

TABLE XXII

Functions for Calculating Resistance and Inductance of Straight, Cylindrical Wires with Varying Frequency (sec. 10)

x	$\frac{W}{Y}$	Δ_1	Δ_2	$\frac{x}{2}\frac{W}{Y}$	Δ_1	Δ_2	$\frac{Z}{Y}$	Δ_1	Δ_2	$\frac{4}{x}\frac{Z}{Y}$	Δ_1	Δ_2
0.0	∞	....	...	1.00000	0	+ 1	0.	+2500	...	1.00000		
.1	20.00000	....	...	1.00000	+ 1	+ 2	0.02500	2500	...	1.00000	0	− 2
.2	10.00020	....	...	1.00001	3	+ 6	0.05000	2500	− 1	1.00000	− 2	− 3
.3	6.66693	....	...	1.00004	9	+ 10	0.07500	2499	0	0.99998	− 5	− 5
.4	5.00065	....	...	1.00013	19	+ 16	0.09999	2499	− 2	0.99993	− 10	− 8
0.5	4.00128	....	...	1.00032	+ 35	+ 22	0.12498	+2497	− 3	0.99984	− 18	−11
.6	3.33557	....	...	1.00067	57	+ 31	0.14995	2494	− 4	0.99966	− 29	−14
.7	2.86069	....	...	1.00124	88	+ 40	0.17489	2490	− 7	0.99937	− 43	−20
.8	2.50530	....	...	1.00212	128	+ 51	0.19979	2483	− 10	0.99894	− 64	−25
.9	2.22978	....	...	1.00340	179	+ 60	0.22462	2473	− 12	0.99830	− 89	−31
1.0	2.01038	....	...	1.00519	+ 239	+ 74	0.24935	+2461	− 18	0.99741	− 120	−36
1.1	1.83196	....	...	1.00758	313	+ 86	0.27396	2443	− 21	0.99621	− 156	−43
1.2	1.68451	....	...	1.01071	399	100	0.29839	2422	− 27	0.99465	− 199	−50
1.3	1.56108	....	...	1.01470	499	114	0.32261	2395	− 34	0.99266	− 249	−57
1.4	1.45670	....	...	1.01969	613	128	0.34656	2361	− 41	0.99017	− 306	−63
1.5	1.36776	....	...	1.02582	+ 741	141	0.37017	+2320	− 48	0.98711	− 369	−69
1.6	1.29154	....	...	1.03323	882	153	0.39337	2272	− 56	0.98342	− 438	−76
1.7	1.22594	....	...	1.04205	1035	165	0.41609	2216	− 64	0.97904	− 514	−81
1.8	1.16934	....	...	1.05240	1200	176	0.43825	2152	− 72	0.97390	− 595	−86
1.9	1.12042	....	...	1.06440	1376	183	0.45977	2080	− 81	0.96795	− 681	−89
2.0	1.07816	−3649	+505	1.07316	1559	192	0.48057	+1999	− 90	0.96113	− 770	−92
2.1	1.04167	−3144	442	1.09375	1751	192	0.50056	1909	− 96	0.95343	− 862	−92
2.2	1.01023	2702	387	1.11126	1943	195	0.51965	1813	−102	0.94482	− 954	−91
2.3	0.98321	2315	339	1.13069	2138	192	0.53778	1711	−108	0.93527	−1045	−90
2.4	0.96006	1976	297	1.15207	2330	188	0.55489	1603	−113	0.92482	−1135	−86
2.5	0.94030	−1679	+256	1.17538	2518	179	0.57092	+1490	−115	0.91347	−1221	−81
2.6	0.92351	1423	224	1.20056	2697	170	0.58582	1375	−116	0.90126	−1301	−73
2.7	0.90928	1199	190	1.22753	2867	157	0.59957	1259	−116	0.88825	−1374	−65
2.8	0.89729	1009	162	1.25620	3024	142	0.61216	1143	−114	0.87451	−1439	−56
2.9	0.88720	847	136	1.28644	3165	128	0.62359	1029	−111	0.86012	−1495	−47
3.0	0.87873	− 711	+114	1.31809	+3293	+109	0.63388	+ 918	−106	0.84517	−1542	−36
3.1	0.87162	597	92	1.35102	3402	+ 93	0.64306	812	−100	0.82975	−1578	−25
3.2	0.86565	505	75	1.38504	3495	+ 76	0.65118	712	− 93	0.81397	−1603	−16
3.3	0.86060	430	58	1.41999	3571	+ 61	0.65830	619	− 87	0.79794	−1619	− 6
3.4	0.85630	372	46	1.45570	3632	+ 44	0.66449	532	− 78	0.78175	−1625	+ 4
3.5	0.85258	− 326	+ 36	1.49202	+3677	+ 31	0.66981	454	− 70	0.76550	−1621	+11
3.6	0.84932	290	24	1.52879	3708	+ 19	0.67436	384	− 61	0.74929	−1610	+20
3.7	0.84642	266	18	1.56587	3727	+ 10	0.67820	323	− 55	0.73320	−1590	26
3.8	0.84376	248	13	1.60314	3737	− 1	0.68143	268	− 47	0.71729	−1564	31
3.9	0.84128	235	8	1.64051	3736	− 7	0.68411	221	− 40	0.70165	−1533	36

TABLE XXII—Continued

x	$\frac{W}{Y}$	Δ_1	Δ_2	$\frac{x}{2}\frac{W}{Y}$	Δ_1	Δ_2	$\frac{Z}{Y}$	Δ_1	Δ_2	$\frac{4}{x}\frac{Z}{Y}$	Δ_1	Δ_2
4.0	0.83893	−227	+5	1.67787	+3729	−12	0.68632	+181	−33	0.68632	−1497	+39
4.1	0.83666	222	3	1.71516	3717	−17	0.68813	148	−28	0.67135	−1458	43
4.2	0.83444	219	+1	1.75233	3700	−19	0.68961	120	−23	0.65677	−1415	43
4.3	0.83225	218	0	1.78933	3681	−20	0.69082	97	−17	0.64262	−1372	45
4.4	0.83007	218	0	1.82614	3661	−22	0.69179	80	−15	0.62890	−1327	45
4.5	0.82789	−218	+1	1.86275	+3639	−20	0.69259	+65	−12	0.61563	−1282	+45
4.6	0.82571	217	+1	1.89914	3619	−21	0.69324	53	−8	0.60281	−1237	45
4.7	0.82354	216	+1	1.93533	3598	−19	0.69377	45	−6	0.59044	−1192	43
4.8	0.82138	215	+1	1.97131	3579	−17	0.69422	39	−4	0.57852	−1149	43
4.9	0.81923	−214	+2	2.00710	+3562	−15	0.69461	35	−3	0.56703	−1106	42
5.0	0.81709	−419	+14	2.04272	+7081	−45	0.69496	+62	−6	0.55597	−2091	+151
5.2	0.81290	405	18	2.11353	7036	−31	0.69558	56	−1	0.53506	1940	138
5.4	0.80885	387	20	2.18389	7005	−19	0.69614	55	0	0.51566	1802	124
5.6	0.80498	367	22	2.25393	6987	−8	0.69669	55	+1	0.49764	1678	112
5.8	0.80131	345	23	2.32380	6979	0	0.69725	56	−1	0.48086	1566	101
6.0	0.79786	−322	+21	2.39359	+6979	+4	0.69781	+55	−1	0.46521	−1465	+91
6.2	0.79464	301	21	2.46338	6983	8	0.69836	54	−2	0.45056	1374	82
6.4	0.79163	280	19	2.53321	6992	8	0.69891	52	−3	0.43682	1292	74
6.6	0.78883	261	17	2.60313	6999	8	0.69942	49	−3	0.42389	1218	68
6.8	0.78621	244	16	2.67312	7007	8	0.69991	46	−4	0.41171	1150	62
7.0	0.78377	−228	+13	2.74319	+7015	+6	0.70037	+42	−3	0.40021	−1088	+57
7.2	0.78149	215	13	2.81334	7021	5	0.70080	39	4	0.38933	1031	52
7.4	0.77934	202	12	2.88355	7026	5	0.70118	35	3	0.37902	979	49
7.6	0.77731	190	11	2.95380	7031	3	0.70154	32	3	0.36923	930	45
7.8	0.77541	179	9	3.02411	7034	+1	0.70185	29	−2	0.35992	885	42
8.0	0.77361	−170	+8	3.09445	+7035	+3	0.70214	+27	−2	0.35107	−843	+39
8.2	0.77191	162	8	3.16480	7038	+1	0.70241	25	−2	0.34263	804	36
8.4	0.77028	154	7	3.23518	7039	+1	0.70265	23	−2	0.33460	768	34
8.6	0.76874	147	7	3.30557	7040	+1	0.70288	20	−1	0.32692	734	32
8.8	0.76727	140	6	3.37597	7041	+1	0.70308	19	−1	0.31958	702	30
9.0	0.76586	−134	+6	3.44638	+7042	+1	0.70327	+18	−2	0.31257	−672	+28
9.2	0.76452	128	5	3.51680	7043	−1	0.70345	16	1	0.30585	644	26
9.4	0.76324	123	6	3.58723	7043	+2	0.70362	15	1	0.29941	617	25
9.6	0.76201	117	4	3.65766	7045	+4	0.70377	14	1	0.29324	593	23
9.8	0.76084	−113	+4	3.72812	7046	+2	0.7039[illegible]	+13	−1	0.28731	−569	+21
10.0	0.75971	−261	+24	3.79857	+17620	+3	[illegible]	+30	−4	0.28162	−1330	+120
10.5	0.75710	237	21	3.97477	17623	4	0.70435	26	4	0.26832	1210	104
11.0	0.75473	216	19	4.15100	17627	4	[illegible]	22	3	0.25622	1106	91
11.5	0.75257	197	16	4.32727	17631	4	[illegible]	19	2	0.24516	1015	80
12.0	0.75060	181	15	4.50358	17635	3	0.70503	17	−2	0.23501	935	71
12.5	0.74879	−166	+13	4.67993	+17638	+3	0.70520	+15	−1	0.22567	−863	+64
13.0	0.74712	154	11	4.85631	17641	2	0.70535	14	2	0.21703	800	57
13.5	0.74559	142	10	5.03272	17643	2	0.70549	12	1	0.20903	743	51
14.0	0.74416	132	9	5.20915	17645	3	0.70561	11	2	0.20160	692	46
14.5	0.74284	−123	+8	5.38560	17648	+2	0.70572	+9	−1	0.19468	−646	+40

TABLE XXII—Continued

x	$\frac{W}{Y}$	Δ_1	Δ_2	$\frac{x}{2}\frac{W}{Y}$	Δ_1	Δ_2	$\frac{Z}{Y}$	Δ_1	Δ_2	$\frac{4}{x}\frac{Z}{Y}$	Δ_1	Δ_2
15.0	0.74161	−222	+27	5.56208	+35301	+ 6	0.70581	+16	−3	0.18822	−1172	+137
16.0	0.73939	195	22	5.91509	35307	5	0.70597	13	2	0.17649	1035	114
17.0	0.73743	173	19	6.26817	35312	5	0.70611	11	1	0.16614	921	97
18.0	0.73570	154	16	6.62129	35317	4	0.70622	10	2	0.15694	824	82
19.0	0.73416	139	13	6.97446	35321	3	0.70632	8	−1	0.14870	742	70
20.0	0.73277	−125	+11	7.32767	+35324	+ 3	0.70640	7	−1	0.14128	− 672	+ 61
21.0	0.73151	114	10	7.68091	35327	2	0.70646	6	1	0.13456	611	53
22.0	0.73038	104	9	8.03418	35329	2	0.70652	5	0	0.12846	558	46
23.0	0.72935	95	8	8.38748	35331	2	0.70657	5	1	0.12288	511	41
24.0	0.72840	87	7	8.74079	35333	2	0.70662	4	−1	0.11777	470	36
25.0	0.72753	− 80	+ 5	9.09412	+35335	+ 2	0.70666	+3	0	0.11307	− 434	+ 32
26.0	0.72673	−143	+20	9.44748	+70674	+ 5	0.70669	6	−1	0.10872	− 776	+103
28.0	0.72530	123	15	10.15422	70679	+ 4	0.70675	5	−1	0.10096	672	84
30.0	0.72407	−108	+13	10.86101	+70683	+ 3	0.70680	+4	−1	0.09424	− 589	+ 69
32.0	0.72299	95	11	11.56785	70686	3	0.70684	3	..	0.08835	519	58
34.0	0.72204	84	9	12.27471	70689	2	0.70687	2	..	0.08316	462	49
36.0	0.72120	75	8	12.98160	70691	2	0.70689	2	..	0.07854	413	41
38.0	0.72045	68	7	13.68852	70693	2	0.70691	2	..	0.07441	372	35
40.0	0.71977	− 61	+ 6	14.39545	+ 70695	1	0.70693	+2	..	0.07069	− 336	+ 30
42.0	0.71916	55	5	15.10240	70696	2	0.70695	1	..	0.06733	306	27
44.0	0.71861	50	4	15.80936	70698	1	0.70696	2	..	0.06427	279	23
46.0	0.71810	46	4	16.51634	70699	0	0.70698	1	..	0.06148	256	20
48.0	0.71764	− 43	+ 3	17.22333	+ 70699	+ 1	0.70699	+1	..	0.05892	− 236	+ 17
50.0	0.71721	−170	+49	17.93032	+353509	+13	0.70700	+3	..	0.05656	− 942	+269
60.0	0.71551	121	30	21.46541	353522	8	0.70703	2	..	0.04713	673	168
70.0	0.71430	91	20	25.00063	353530	5	0.70705	1	..	0.04040	505	112
80.0	0.71340	70	+14	28.53593	353535	+ 3	0.70706	1	..	0.03535	393	79
90.0	0.71270	− 56	..	32.07127	353538	..	0.70707	+1	..	0.03142	− 314	+ 55
100.0	0.71213			35.60666			0.70708			0.02828		
∞	0.70711			∞			0.70711			0.		

TABLE XXIII

Values of Limiting Change of Inductance with the Frequency

$\frac{2l}{\rho}$	Single Wire			$\frac{d}{\rho}$	Parallel Wires			$\frac{8a}{\rho}$	Circular Rings		
	$\left(\frac{\Delta L}{L}\right)_{x=\infty}$	Δ_1	Δ_2		$\left(\frac{\Delta L}{L}\right)_{x=\infty}$	Δ_1	Δ_2		$\left(\frac{\Delta L}{L}\right)_{x=\infty}$	Δ_1	Δ_2
50	0.07906			5	0.13445	−1201	+ 342				
				6	.12244	859	206				
100	0.06485	−988	+538	7	.11385	653	137	100	0.08756	−1710	+987
200	5497	450	173	8	.10732	516	84	200	7046	723	294
300	5047	277	82	9	.10216	− 432		300	6323	429	134
400	4770	195	47					400	5894	295	75
500	4575	−148	+ 30	10	0.09784	−2078	+1219	500	5600	− 220	+ 47
				20	7706	859	359				
600	0.04427	−118	+ 21	30	6847	500	160	600	0.05380	− 173	+ 32
700	4309	97	15	40	6347	340	88	700	5207	141	23
800	4213	82	11	50	6007	− 252	+ 55	800	5066	118	17
900	4131	− 71	+ 9					900	4948	− 101	+ 13
				60	0.05755	− 197	+ 37				
1000	0.04060	−411	+207	70	5557	160	26	1000	0.04847	− 574	+297
2000	3649	204	72	80	5397	134	20	2000	4273	277	101
3000	3445	131	36	90	5263	− 114	+ 15	3000	3996	176	50
4000	3314	95	22					4000	3820	126	29
5000	3219	− 74	+ 14	100	0.05149	− 643	+ 336	5000	3694	− 97	+ 19
				200	4506	307	113				
6000	0.03145	− 60	+ 10	300	4199	194	56	6000	0.03597	− 78	+ 13
7000	3085	50	7	400	4005	138	32	7000	3519	65	10
8000	3035	43	6	500	3867	− 106	+ 21	8000	3454	55	8
9000	2992	− 37	+ 4					9000	3399	− 48	+ 6
				600	0.03761	− 85	+ 14				
10000	0.02955	−224	+108	700	3676	71	11	10000	0.03351	− 285	+140
20000	2731	116	40	800	3605	60	8	20000	3066	145	50
30000	2615	76	20	900	3545	− 52	+ 6	30000	2921	95	25
40000	2539	56	12					40000	2826	70	16
50000	2483	− 44	+ 8	1000	0.03493	− 309	+ 154	50000	2756	− 54	+ 10
				2000	3183	155	53				
60000	0.02438	− 36	+ 6	3000	3028	102	27	60000	0.02702	− 44	+ 7
70000	2402	30	4	4000	2926	75	17	70000	2658	37	5
80000	2372	26	3	5000	2851	− 58	+ 11	80000	2621	32	4
90000	2346	− 23	+ 2					90000	2589	− 28	+ 3
				6000	0.02793	− 47	+ 7				
100000	0.02323			7000	2746	40	6	100000	0.02561		
1000000	1913			8000	2706	34	4	1000000	2072		
				9000	2672	− 30	+ 3				
				10000	2643						

TABLE XXIV

Values of the Argument x_0 for Copper Wires 1 mm Radius and Conductivity $5.811 x 10^{-4}$ c g. s. Units

f cycles per second	x_0	λ meters	f cycles per second	x_0	λ meters
100	0.2142	....	50000	4.790	6000
200	.3029	....	60000	5.247	5000
300	.3710	....	70000	5.667	4286
400	.4284	....	80000	6.058	3750
500	.4790	....	90000	6.426	3333
600	0.5247	....	100000	6.774	3000
700	.5667	....	150000	8.296	2000
800	.6058	....	200000	9.579	1500
900	.6426	....	250000	10.710	1200
1000	.6774	....	300000	11.732	1000
2000	0.9579	....	333333	12.367	900
3000	1.1732	....	375000	13.117	800
4000	1.3547	....	428570	14.023	700
5000	1.5146	....	500000	15.146	600
			600000	16.592	500
6000	1.6592	....	700000	17.921	429
7000	1.7921	....	750000	18.550	400
8000	1.9158	....	800000	19.158	375
9000	2.0321	....	900000	20.321	333
....	....	....	1000000	21.42	300
10000	2.142	30000	1500000	26.23	200
15000	2.623	20000	3000000	37.10	100
20000	3.029	15000			
30000	3.710	10000			
40000	4.284	7500			

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[Italicized page numbers refer to examples illustrating the formulas. Proper names are also italicized.]

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